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Solution of the Boolean Markus–Yamabe Problem

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The main point in the design of *content addressable memory* would be under what conditions the state possessing the total information can attract all other states in the phase of the system. The problem can be formulated as a global asymptotic stability problem of Boolean dynamical systems. In this article we give a complete answer to this global asymptotic stability problem. The conditions employed involve the *Hamming distance* on the phase space $\{0, 1\}^n$ as well as the spectral condition on the *Jacobian Boolean matrix* of F: $\{0, 1\}^n \to \{0, 1\}^n$ evaluated at each point of $\{0, 1\}^n$. This article furnishes a complete solution of the *Boolean Markus–Yamabe problem.*

Key Words: content addressable memory; Boolean matrix; Boolean dynamical system; global attractor; eigenvalue; spectral radius; von Neumann neighborhood; Hamming distance; Markus–Yamabe conjecture; automata network.

1. INTRODUCTORY REMARKS

The efficiency of a *content addressable memory* (CAM) depends on whether the information flow converges to the state containing the whole information, provided the initial state which contains partial information is chosen from the region preassigned arbitrarily, see Hopfield [14]. Thus the

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main point in the design of CAM would be under what conditions the state possessing the total information can attract all other states in the phase of the system. The mathematical problem associated can be described as the following:

Problem. Let F: $\{0,1\}^n \to \{0,1\}^n$. Consider the network,

$$F(x^r) = x^{r+1}, \qquad r = 0, 1, \ldots, \qquad x^0 \in \{0,1\}^n.$$

Under what conditions on F does F have a unique fixed point ξ and a positive integer $p\,(\leq 2^n)$ such that the pth iterate $F^p(x^0) = x^p = \xi$ for any initial $x^0 \in \{0,1\}^n$?

Let us recall that a *fixed point* of F is a point x such that $F(x) = x$. The preceding problem may be regarded as a global asymptotic stability problem of Boolean dynamical systems. Local asymptotic stability problems of Boolean dynamical systems, proposed and explored by Robert in [22] and [23], may be stated as follows: If F: $\{0,1\}^n \to \{0,1\}^n$ with $F(\xi) = \xi$, under what conditions on F does there exist a positive integer $p \leq n$ such that for any x in the von Neumann neighborhood V_ξ of ξ, the iterates $F^k(x) \in V_\xi$ $(k = 1, 2 \cdots)$ and $F^p(x) = \xi$? Robert in [22] and [23] developed a systematic theory which solved this local asymptotic stability problem (see Theorem 2.7 of Section 2) as well as a variety of diverse discrete iteration problems. The object of this article will be to give a complete answer to the preceding global asymptotic problem. Our conditions presented here are related to the *Hamming distance* on the phase space $\{0,1\}^n$ as well as to the *discrete spectral condition* of the *Markus–Yamabe conjecture* in differential equations [17]. The content of this paper is organized as follows. Section 2 is a development of the tools, especially the spectra of Boolean matrices, needed to prove the main theorems. Section 3 is a presentation of two global asymptotic stability theorems. Sections 4 and 5 are primarily devoted to the proofs of these two global asymptotic stability theorems, respectively. This article ends with a conjecture in Section 6.

Let us remark that our main concern of this article may be regarded as the *Boolean Markus–Yamabe problem.* The main results of this article give a complete solution of this *Boolean Markus–Yamabe problem.*

2. CONCERNING THE SPECTRA OF BOOLEAN MATRICES

In this section, we state some notions and results concerning the spectra of Boolean matrices needed to formulate and to prove the main results. The material can be found in the fundamental paper by Robert [22] and also in the book by Robert [23] (see also Brualdi and Ryser [5] for a combinatorial theory of (0, 1)-matrices).

Let $\{0,1\}$ be with three operations $+, \cdot, \bar{}$ defined as follows,

$$0+0=0\cdot 1=1\cdot 0=0\cdot 0=0, \qquad 1+0=0+1=1+1=1\cdot 1=1,$$
$$\bar{0}=1, \quad \text{and} \quad \bar{1}=0.$$

For $a, b \in \{0,1\}$, we usually suppress the dot "$\cdot$" of a, b and simply write ab. For each positive integer n, let $\{0,1\}^n$ be the set of ordered n-tuples,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with components $x_i \in \{0,1\}$ $(i=1,\ldots,n)$. We also write $x=(x_1,\ldots,x_n)$ interchangably. We may think x as a *bit string* of length n, thus we may write $x = x_1x_2\cdots x_n$. The zero element of $\{0,1\}^n$ is the point $\mathbf{0}$, all of whose coordinates are 0. The order "$\le$" in $\{0,1\}$ is given by $0 \le 0 \le 1 \le 1$. Thus for $a, b \in \{0,1\}$,

$$a+b=\max\{a,b\}, \qquad ab=\min\{a,b\}.$$

For $x, y \in \{0,1\}^n$, $x \le y$ is meant that $x_i \le y_i$ $(i=1,\ldots,n)$. For $x, y \in \{0,1\}^n$ and $\lambda \in \{0,1\}$, define

$$x+y \overset{\text{def}}{=} \begin{pmatrix} \max\{x_1,y_1\} \\ \vdots \\ \max\{x_n,y_n\} \end{pmatrix}, \qquad \lambda x \overset{\text{def}}{=} \begin{pmatrix} \min\{\lambda,x_1\} \\ \vdots \\ \min\{\lambda,x_n\} \end{pmatrix}.$$

Throughout this article, a *Boolean matrix* is meant to be a matrix over $\{0,1\}$. Boolean matrix addition and Boolean matrix multiplication are the same as in the case of complex matrices but the concerned sums and products of entries are Boolean. Let $F\colon \{0,1\}^n \to \{0,1\}^n$ and let us write $F=(f_1,\ldots,f_n)$. According to Robert ([23, p. 7]), the *incidence matrix* of F is the $n\times n$ Boolean matrix defined by

$$B(F)=(b_{ij}),$$

where $b_{ij} \overset{\text{def}}{=} 0$ if f_i does not depend on x_j, $b_{ij} \overset{\text{def}}{=} 1$ otherwise. More precisely, $b_{ij}=0$ if for any fixed $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n$, $f_i(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)$ is a constant function of x_j on $\{0,1\}$, $b_{ij}=1$ otherwise. Throughout this article, A is always denoted as an $n\times n$ Boolean matrix. A nonzero element $u \in \{0,1\}^n$ is called a (Boolean) eigenvector of A if there exists λ in $\{0,1\}$ such that $Au=\lambda u$; λ is called the (Boolean) *eigenvalue* associated with the eigenvector u. The symbol $\sigma(A)$ stands for the set of all (Boolean) eigenvalues of A, so that $\sigma(A) \subset \{0, 1\}$. The *Boolean spectral radius* of A, which is denoted by $\rho(A)$, is defined to be the largest (Boolean) eigenvalues of A. Because $\sigma(A) \neq \phi$ (this fact is not a priori obvious, see [22, p. 48]), $\rho(A) = 0$ or 1. Also $\rho(P^t AP) = \rho(A)$ for any *permutation* matrix P. For $x \in \{0, 1\}^n$, let

$$\tilde{x}^j \overset{\text{def}}{=} \begin{pmatrix} x_1 \\ \vdots \\ \bar{x}_j \\ \vdots \\ x_n \end{pmatrix}.$$

The notation $\tilde{x}_i^j$ is meant that $\tilde{x}_i^j = \bar{x}_j$ if $i = j$, $\tilde{x}_i^j = x_i$ if $i \neq j$. Because we may identify $\{0, 1\}^n$ with the vertices of the n-cube, $\tilde{x}^j$ may be interpreted as the *j-neighbour* of x ($j = 1, \dots, n$). For $x \in \{0, 1\}^n$, the *von Neumann neighbourhood* of x ([12, p. 17]) is defined to be the set V_x of vertices of the n-cube formed by x and its n neighbours, that is,

$$V_x \overset{\text{def}}{=} \{x, \tilde{x}^1, \dots, \tilde{x}^n\}.$$

For the sake of simplicity, let us introduce the following notations. For $x \in \{0, 1\}^n$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, n\}$, let us define $\tilde{x}^{j_1,\dots,j_k} = y$ by

$$y_i \overset{\text{def}}{=} \begin{cases} x_i, & \text{if } i \neq j_1, \dots, j_k, \\ \bar{x}_i, & \text{if } i = j_1, \dots, j_k. \end{cases}$$

If $\Lambda = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$, and $i \in \{1, \dots, n\}$, let us write $\tilde{x}^\Lambda$ for $\tilde{x}^{j_1,\dots,j_k}$ and let us write $\tilde{x}^{i,\Lambda}$ for $\tilde{x}^{i,j_1,\dots,j_k}$. For $i \in \{1, \dots, n\}$ and $\alpha(i) \in \{0, 1\}^n$, denote the j-component of $\alpha(i)$ by $\alpha(i)_j$ ($j = 1, \dots, n$).

The *discrete derivative* (or the *Jacobian Boolean matrix*) of F at $x \in \{0, 1\}^n$ is the (Boolean) $n \times n$ matrix defined by

$$F'(x) = (f_{ij}(x)),$$

where $f_{ij}(x) \overset{\text{def}}{=} 1$ if $f_i(x) \neq f_i(\tilde{x}^j)$, $f_{ij}(x) \overset{\text{def}}{=} 0$ otherwise. The discrete metric on $\{0, 1\}$ is denoted by δ, that is, $\delta(x, y) = 1$ if $x \neq y$, $\delta(x, y) = 0$ if $x = y$. For $x, y \in \{0, 1\}^n$, *the Boolean vector distance* $d(x, y)$ is defined by

$$d(x, y) \overset{\text{def}}{=} \begin{pmatrix} \delta(x_1, y_1) \\ \vdots \\ \delta(x_n, y_n) \end{pmatrix}.$$

Let us recall that the Boolean vector distance d satisfies:

(i) $d(x, y) = d(y, x)$ $(x, y \in \{0, 1\}^n)$,

(ii) $d(x, y) = \mathbf{0} \iff x = y$ $(x, y \in \{0, 1\}^n)$,

(iii) $d(x, y) \le d(x, z) + d(z, y)$ $(x, y, z \in \{0, 1\}^n)$,

where $d(x, z) + d(z, y)$ is the Boolean sum in $\{0, 1\}^n$.

Let us recall some elementary graph-theoretic notions. The *digraph* (*directed graph*) of an $n \times n$ Boolean matrix $A = (a_{ij})$, denoted by $\Gamma(A)$, is the digraph on n nodes $P_1, \dots, P_n$ such that there is a directed arc from P_i to P_j if and only if $a_{ij} = 1$. A *directed path* in $\Gamma(A)$ is a sequence of directed arcs $P_{i_1}P_{i_2}$, $P_{i_2}P_{i_3}, \dots$ in $\Gamma(A)$. A *cycle* in $\Gamma(A)$ is a directed path that begins and ends at the same node. The *length* of a directed path in $\Gamma(A)$ is the number of successive directed arcs in the directed path.

We now state some basic results concerning the spectral theory of Boolean matrices.

THEOREM 2.1. *The following conditions are mutually equivalent*:

(i) $\rho(A) = 1$.

(ii) *A contains a principal submatrix which has no zero rows.*

(iii) *A contains a principal submatrix which has no zero columns.*

(iv) *$\Gamma(A)$ contains a cycle.*

The following result, a direct consequence of Theorem 2.1, is also useful for the determination of the Boolean spectral radius which equals 1.

THEOREM 2.2. *If $\rho(A) = 0$, then A has a zero column and a zero row. Furthermore, each entry in the diagonal of A is* 0.

THEOREM 2.3. *The following conditions are mutually equivalent*:

(a) $\rho(A) = 0$.

(b) *There exists a permutation matrix P such that $P^t AP$ is strictly upper triangular.*

(c) *There exists a positive integer $p \le n$ such that $A^p = 0$.*

A proof of the equivalence (a) $\iff$ (c) runs as follows: If $A^n \ne 0$, then $\Gamma(A)$ contains a directed path of length n. By the *pigeon-hole principle*, $\Gamma(A)$ contains a cycle, so that $\rho(A) = 1$ by Theorem 2.1. Conversely, if $A^n = 0$, then $\rho(A) = \rho(A^n) = 0$.

Let us recall that a map $F\colon \{0, 1\}^n \to \{0, 1\}^n$ is said to be a *contraction with respect to the boolean vector distance d* (or simply a *contraction*) if there is an $n \times n$ Boolean matrix M having $\rho(M) = 0$ such that

$$d(F(x), F(y)) \le Md(x, y), \qquad (x, y \in \{0, 1\}^n).$$

It is proved that F is a contraction if and only if $\rho(B(F)) = 0$ (see [23, p. 58]).

THEOREM 2.4. *If F: $\{0,1\}^n \to \{0,1\}^n$ is a contraction, then F has a unique fixed point and there exists a positive integer $p \le n$ such that $F^p(x) = \xi$ for all $x \in \{0,1\}^n$.*

Let us remark that Theorem 2.4 amounts to saying that if the digraph $\Gamma(B(F))$ contains no cycles, then the *iteration graph* for F is *simple*. Recall that the *iteration graph* for F is the digraph consisting of vertices which are elements of $\{0,1\}^n$ and the following directed arcs: for all x in $\{0,1\}^n$, a directed arc connects x to $F(x)$. The iteration graph for F is said to be *simple* if the iteration graph has only one *basin* and this basin has a unique fixed point for F.

THEOREM 2.5. *Let F: $\{0,1\}^n \to \{0,1\}^n$. Then*

(a) $F'(x) \le B(F)$ $(x \in \{0,1\}^n)$,

(b) $B(F) = \sup_{x\in\{0,1\}^n}\{F'(x)\}$, *where the order and the* sup *are taken elementwise on the Boolean matrices of size $n \times n$ using the order* $0 \le 0 \le 1 \le 1$ *on* $\{0,1\}$.

THEOREM 2.6. *Let F: $\{0,1\}^n \to \{0,1\}^n$. Then*

$$d(F(x), F(y)) = F'(x)d(x,y), \qquad (x \in \{0,1\}^n,\ y \in V_x),$$

where $d(x,y)$ is the Boolean vector distance on $\{0,1\}^n$.

THEOREM 2.7 (Robert). *Let F: $\{0,1\}^n \to \{0,1\}^n$ with $F(\xi) = \xi$. If $\rho(F'(\xi)) = 0$ and $F'(\xi)$ has at most 1 in each column, then $F(V_\xi) \subset V_\xi$ and there exists a positive integer $p \le n$ such that $F^p(x) = \xi$ for any $x \in V_\xi$.*

As pointed out by Robert ([23, pp. 107–108]; see also Robert [21]), Theorem 2.7 has a pertinent interpretation in the context of *automata networks*. Let us remark that in Theorem 2.7 the condition "$\rho(F'(\xi)) = 0$" is by itself not sufficient to guarantee the ξ is *attractive* in all of V_ξ (see Robert [23, pp. 105–106]). Concerning Theorem 2.7, remark also that the condition "*$F'(\xi)$ has at most* 1 *in each column*" is equivalent to the condition "*$F(V_\xi) \subset V_\xi$*" (see Robert [23, pp. 103–104, proof of Theorem 4]; see also Lemma 4.1 of Section 4 of this article.) It may be observed here that by virtue of Theorems 2.3 and 2.6, the sequence $\{F^k(x)\}$ in the von Neumann neighborhood V_ξ reaches the fixed point ξ in *at most n* steps. Finally we want to refer to the fundamental paper of J. von Neumann [24] which contains a very valuable source of information about cellular automata theory.

3. GLOBAL ASYMPTOTIC STABILITY THEOREMS

Let F: $\{0,1\}^n \to \{0,1\}^n$. Consider the network,

$$F(x^r) = x^{r+1}, \qquad r = 0,1,\ldots, \qquad x^0 \in \{0,1\}^n. \tag{3.1}$$

We say that the F is *simple* (or the iteration graph for F is *simple*) if F has a unique fixed point ξ and ξ is a global attractor for (3.1); i.e., there exists a positive integer $p\,(\leq 2^n)$ such that $F^p(x^0) = x^p = \xi$ for any initial $x^0 \in \{0,1\}^n$. Equivalently, F is simple if there exists $\xi \in \{0,1\}^n$ such that for any initial $x^0 \in \{0,1\}^n$, there is a positive integer $p(x^0)\,(\leq 2^n)$ so that $F^{p(x^0)}(x^0) \equiv \xi$.

This article will be devoted to the proofs of the following two theorems.

THEOREM 3.1. *Let* $F\colon \{0,1\}^n \to \{0,1\}^n$. *Suppose*

(a) $\rho(F'(x)) = 0$ *for all* x *in* $\{0,1\}^n$,

(b) $F(V_x) \subset V_{F(x)}$ *for all* x *in* $\{0,1\}^n$.

Then F *is simple* (*for all* n).

THEOREM 3.2. *Let* $F\colon \{0,1\}^n \to \{0,1\}^n$ *with* $\rho(F'(x)) = 0$ *for all* x *in* $\{0,1\}^n$. *Then*

(a) *if* $n \leq 3$, F *is simple* (*via* F *is a contraction for* $n = 1$ *and* 2 *but not necessarily a contraction for* $n = 3$),

(b) *if* $n > 3$, F *is not necessarily simple even if* F *admits a fixed point*. *More precisely*, *there is a map* $F\colon \{0,1\}^n \to \{0,1\}^n$, $n > 3$, *with* $F(\xi) = \xi$ *such that a cycle of* F *can occur.*

Let us first recall that the *Hamming metric* ρ_{H} ([15, p. 64]) on $\{0,1\}^n$ is defined by

$$\rho_{\mathrm{H}}(x, y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \delta(x_i, y_i), \qquad (x, y \in \{0,1\}^n).$$

Hence $\rho_{\mathrm{H}}((0,0,0,0,0,0),\ (0,1,0,1,0,1)) = 3$. From a metric-theoretic view point, condition (b) of Theorem 3.1 can be formulated as follows (see Lemma 4.3),

$$\text{“}\rho_{\mathrm{H}}(F(x), F(y)) \leq \rho_{\mathrm{H}}(x, y), \qquad (x, y \in \{0,1\}^n).\text{”}$$

From a matrix-theoretic view point, condition (b) of Theorem 3.1 can be formulated as follows (see Lemma 4.1),

“For each $x \in \{0,1\}^n$, $F'(x)$ has at most 1 in each column.”

Let us note that condition (a) of Theorem 3.1 can be formulated as follows (see Theorem 2.1):

“For each $x \in \{0,1\}^n$ and for each principal submatrix $H(x)$ of $F'(x)$, $H(x)$ has a zero row or a zero column.”

or

“For each $x \in \{0,1\}^n$, the digraph $\Gamma(F'(x))$ contains no cycles.”

The conditions given in Theorems 3.1 and 3.2 have the following logical relationships. Let

$$P\colon \rho(F'(x)) = 0, \quad \text{for all } x \in \{0,1\}^n,$$
$$Q\colon F(V_x) \subset V_{F(x)}, \quad \text{for all } x \in \{0,1\}^n,$$
$$R\colon F\colon \{0,1\}^n \to \{0,1\}^n \text{ is a contraction.}$$

Then we have

(1) $R \Longrightarrow P$ $(n \geq 1)$,

(2) $R \Longleftrightarrow P$ $(n = 1, 2)$,

(3) $P \not\Longrightarrow R$ $(n \geq 3)$,

(4) $P \wedge Q \Longrightarrow R$ $(n = 3)$,

(5) $P \wedge Q \not\Longrightarrow R$ $(n > 3)$,

(6) $R \Longrightarrow Q$ $(n = 1, 2)$,

(7) $R \not\Longrightarrow P \wedge Q$ $(n \geq 3)$.

Let us explain the previous implications in order.

(1) is immediate because by Theorem 2.5,

$$F'(x) \leq B(F), \quad \text{for all } x \in \{0,1\}^n,$$

and by the Boolean Perron–Frobinus theorem ([23, p. 51]),

$$\rho(F'(x)) \leq \rho(B(F)), \quad \text{for all } x \in \{0,1\}^n.$$

(2) The case $n = 1$ holds trivially. The proof of the case $n = 2$ will be given in Section 5.

(3) Let $F\colon \{0,1\}^3 \to \{0,1\}^3$ be defined by

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} x_2 x_3 \\ 1 \\ x_1 + x_2 \end{pmatrix}, \qquad (x \in \{0,1\}^3).$$

Then F is given by Table 1. By Theorem 2.1, it is readily seen that $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^3$ and $\rho(B(F)) = 1$.

(4) Suppose, on the contrary, that $\rho(B(F)) = 1$. Then by Theorem 2.1, the digraph $\Gamma(B(F))$ contains a cycle of length ℓ $(1 \leq \ell \leq 3)$. We divide the proof into three cases $\ell = 1$, $\ell = 2$, and $\ell = 3$ separately, and we show

TABLE 1

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	010	010	011	111	011	011	011	111

that each case arrives at a contradiction. Because the detailed proof seems unncessary, we illustrate only two subcases of $\ell = 2$; the proof of other cases are quite similar.

Now suppose $\ell = 2$. Then $b_{12} = b_{21} = 1$ or $b_{13} = b_{31} = 1$ or $b_{23} = b_{32} = 1$. Let $B(F) = (b_{ij})$. By condition P and Theorem 2.5, we see that

$$b_{11} = b_{22} = b_{33} = 0. \tag{3.2}$$

Suppose $b_{12} = b_{21} = 1$ and $b_{13} = b_{31} = b_{32} = 0$. Then by (3.2),

$$B(F) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.5, we see that

$$\begin{aligned} f_{11}(x) &= f_{13}(x) = f_{22}(x) = f_{23}(x) \\ &= f_{31}(x) = f_{32}(x) = f_{33}(x), \quad \text{for all } x \in \{0,1\}^3. \end{aligned} \tag{3.3}$$

Claim. "$F'(x) = F'(y)$ $(x, y \in \{0,1\}^3)$."

By (3.3), we need only to prove that

$$f_{21}(x) = f_{21}(y) \quad \text{and} \quad f_{12}(x) = f_{12}(y), \quad \text{for all } x, y \in \{0,1\}^3.$$

By $f_{22}(x) = f_{23}(x) = 0$ for all $x \in \{0,1\}^3$, we have

$$f_2(0,0,0) = f_2(0,1,0) = f_2(0,0,1) = f_2(0,1,1),$$
$$f_2(1,0,0) = f_2(1,1,0) = f_2(1,0,1) = f_2(1,1,1).$$

Thus,

$$\begin{aligned} f_{21}(0,0,0) = f_{21}(0,1,0) = f_{21}(0,0,1) = f_{21}(0,1,1) \\ f_{21}(1,0,0) = f_{21}(1,1,0) = f_{21}(1,0,1) = f_{21}(1,1,1). \end{aligned} \tag{3.4}$$

By Lemma 5.2 (see Section 5), $F'(0,0,0)$ and $F'(1,0,0)$ have the same first row; thus,

$$f_{21}(0,0,0) = f_{21}(1,0,0). \tag{3.5}$$

Hence (3.4) and (3.5) together imply that

$$f_{21}(x) = f_{21}(y), \quad \text{for all } x, y \in \{0,1\}^3.$$

Similarly, we can prove that $f_{12}(x) = f_{12}(y)$ for all $x, y \in \{0,1\}^3$. This proves the claim.

By Theorem 2.5 and the claim, we see that

$$F'(x) = B(F), \quad \text{for all } x \in \{0,1\}^3,$$

so that $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^3$, which contradicts condition P.

Suppose $b_{12} = b_{21} = b_{13} = b_{23} = 1$, and $b_{31} = b_{32} = 0$. Then by (3.2), we have

$$B(F) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

For a 3×3 Boolean matrix $A = (a_{ij})$, we write

$$A_{\{2,3\}} \overset{\text{def}}{=} \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}.$$

By Lemma 5.1 (see Section 5), we see that

$$\begin{aligned} &F'(0,0,0)_{\{2,3\}} = F'(0,1,0)_{\{2,3\}} = F'(0,0,1)_{\{2,3\}} = F'(0,1,1)_{\{2,3\}}, \\ &F'(1,0,0)_{\{2,3\}} = F'(1,1,0)_{\{2,3\}} = F'(1,0,1)_{\{2,3\}} = F'(1,1,0)_{\{2,3\}}. \end{aligned} \tag{3.7}$$

By Lemma 5.2, we see that $F'(x)$ and $F'(\tilde{x}^1)$ have the same first row. Thus by (3.6), (3.7), and Theorem 2.5, we can conclude that there is $x \in \{0,1\}^3$ such that

$$f_{13}(x) = f_{23}(x) = 1,$$

which contradicts condition Q.

(5) Let F: $\{0,1\}^4 \to \{0,1\}^4$ be defined by

$$F(x) \overset{\text{def}}{=} \begin{pmatrix} \bar{x}_2 + \bar{x}_3 + x_4 \\ 1 \\ 1 \\ \bar{x}_1 + x_2 + x_3 \end{pmatrix}, \qquad (x \in \{0,1\}^4).$$

Then F is given by Tables 2 and 3. A computation for the discrete derivatives $F'(x)$ of F evaluated at each point $x \in \{0,1\}^4$ now shows that F

TABLE 2

Bit string x	0000	1000	0100	0010	0001	1100	1010	1001
Bit string $F(x)$	1111	1110	1111	1111	1111	1111	1111	1110

TABLE 3

Bit string x	0110	0101	0011	1110	1101	1011	0111	1111
Bit string $F(x)$	0111	1111	1111	0111	1111	1111	1111	1111

satisfies conditions P and Q. Because

$$B(F) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$\rho(B(F)) = 1$; i.e., F is *not* a contraction.

(6) Trivial.

(7) Let F: $\{0,1\}^3 \to \{0,1\}^3$ be defined by

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} x_2 x_3 \\ 1 \\ x_2 \end{pmatrix}, \qquad (x \in \{0,1\}^3).$$

Then F is given by Table 4. Then $\rho(B(F)) = 0$, but $F(V_{(0,0,1)}) \not\subset V_{F(0,0,1)}$ because the second column of

$$F'(0,0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

contains two 1s.

Two further remarks concerning Theorems 3.1 and 3.2 are in order. First, in Theorem 3.1, condition (b) alone does not imply the existence of a fixed point of F, as the following example illustrates. It is of interest to looking for an adequate general fixed point theory on $\{0,1\}^n$. Let us mention that the Hopf–Lefschetz fixed point theorem for order-preserving maps on a finite poset was developed and proved by Baclawski–Björner [1] (see also Baclawski [2]).

EXAMPLE 3.1. Let F: $\{0,1\}^2 \to \{0,1\}^2$ be defined by

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} x_2 \\ \overline{x}_1 \end{pmatrix}, \qquad (x \in \{0,1\}^2).$$

Then F is given by Table 5. Therefore $F(V_x) \subset V_{F(x)}$ for all x in $\{0,1\}^2$, but F has no fixed points.

TABLE 4

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	010	010	011	111	010	010	011	111

TABLE 5

Bit string x	00	10	01	11
Bit string $F(x)$	01	00	11	10

TABLE 6

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	000	000	100	100	010	010	000	000

Second, if condition (b) of Theorem 3.1 is satisfied and if F has a unique fixed point, then the F is *not necessarily* simple. The following example illustrates this.

EXAMPLE 3.2. Let $F\colon \{0,1\}^3 \to \{0,1\}^3$ be defined by

$$F(x) = \begin{pmatrix} \overline{x}_1 x_2 \\ x_1 \overline{x}_2 \\ 0 \end{pmatrix}, \qquad (x \in \{0,1\}^3).$$

Then F is given by Table 6. From Table 6, we see that $F(V_x) \subset V_{F(x)}$ for all x in $\{0,1\}^3$ and F has a unique fixed point $\mathbf{0}$. But $\mathbf{0}$ is not a global attractor because $F(0,1,0) = (1,0,0)$ and $F(1,0,0) = (0,1,0)$.

We close this section with a comparison to the development on the Markus–Yamabe conjecture [17].

Consider a real n-dimensional $\mathscr{C}^1$ vector field T that vanishes at the origin. The Markus–Yamabe conjecture states that if the eigenvalues of the Jacobian matrix $JT(x)$ of T evaluated at each point $x \in \mathbf{R}^n$ have negative real part, then the origin is globally asymptotically stable.

A particular case of the Markus–Yamabe conjecture is the so-called Kalman conjecture, see [3]. The Markus–Yamabe conjecture was solved affirmatively in the case $n = 2$ for polynomial vector fields by Meisters and Olech in 1988 [18]. In the same year Barabanov published a paper [3] containing ideas to construct a $\mathscr{C}^1$-counterexample to the Kalman conjecture for $n \geq 4$ and so to the Markus–Yamabe conjecture. In fact, in 1994, such a counterexample even analytic, was constructed by Bernet and Llibre [4]. In 1993 the Markus–Yamabe conjecture was completely solved affirmatively for $n = 2$ independently by Fessler in [9] and Gutierrez in [13]. In 1994 a much simpler proof of the case $n = 2$ was given by Glutsyuk [11]. In 1997, *a polynomial counterexample* in dimension 3 was constructed by Cima *et al.* [7]. Therefore the Markus–Yamabe conjecture has been completely solved!

Furthermore, a discrete version of the Markus–Yamabe conjecture was proposed by Cima, Gasull, and Mañosas [6]; i.e., let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial map with $T(\xi) = \xi$ and such that the spectral radius $\rho(JT(x)) < 1$ for all $x \in \mathbf{R}^n$, does it follow that for each $x \in \mathbf{R}^n$, $T^k(x)$ tends to ξ if k tends to infinity? It was shown in Cima, Gasull, and Mañosas [6] that the answer is affirmative if $n = 2$. However, in 1997, van den Essen and Hubbers [8] gave a family of counterexamples to this question for $n \geq 4$. In 1997, Cima, *et al.* [7] gave a counterexample to the polynomial discrete Markus–Yamabe problem for all $n \geq 3$. Therefore the polynomial discrete Markus–Yamabe problem has been completely solved! On the other hand, LaSalle in ([16, pp. 21–22]) proposed the $\mathscr{C}^1$ *discrete Markus–Yamabe conjecture* which states that if T: $\mathbf{R}^n \to \mathbf{R}^n$ is a $\mathscr{C}^1$ map with $T(\xi) = \xi$ and such that the spectral radius $\rho(JT(x)) < 1$ for all $x \in \mathbf{R}^n$, does it follow that for each $x \in \mathbf{R}^n$, $T^k(x)$ tends to ξ if k tends to infinity? Evidently, the example given by Cima *et al.* [7] provides a counterexample to the $\mathscr{C}^1$ discrete Markus–Yamabe problem for $n \geq 3$. Szlenk (see [6]) gave a counterexample to the $\mathscr{C}^1$ discrete Markus–Yamabe problem for $n = 2$. Thus the $\mathscr{C}^1$ discrete Markus–Yamabe problem has been completely solved!

Our main concern of this article may be called the *Boolean Markus–Yamabe problem* which is stated as follows: Let F: $\{0, 1\}^n \to \{0, 1\}^n$ with $F(\xi) = \xi$ and such that $\rho(F'(x)) = 0$ for all $x \in \{0, 1\}^n$, does it follow that there is a positive integer $p\,(\leq 2^n)$ such that $F^p(x) = \xi$ for any $x \in \{0, 1\}^n$? Theorems 3.1 and 3.2 give a complete solution of the Boolean Markus–Yamabe problem.

Finally let us observe that one of the main points of the Markus–Yamabe conjecture is a study whether "the collective local effect can yield a global effect." The theorem of Gale and Nikaido in [10] (see also Nikaido [19, p. 370]) concerning the "global inverse function theorem" provides also a famous example of such a phenomenon.

4. PROOF OF THEOREM 3.1

In order to establish Theorem 3.1 we shall employ the following lemmas. As usual, let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

LEMMA 4.1. *Let F: $\{0, 1\}^n \to \{0, 1\}^n$. Then the following conditions are mutually equivalent*:

(a) $F(V_x) \subset V_{F(x)}$ *for all* $x \in \{0, 1\}^n$.

(b) $F'(x)$ *has at most* 1 *in each column for all* $x \in \{0, 1\}^n$.

(c) $F^m(V_x) \subset V_{F^m(x)}$ $(m = 1, 2, \ldots)$ *for all* $x \in \{0, 1\}^n$.

Proof. (a) $\Rightarrow$ (b). Suppose there exists $x \in \{0, 1\}^n$ such that the jth column of $F'(x)$ contains several 1s. Then by Theorem 2.6,

$$\begin{aligned} d(F(\tilde{x}^j), F(x)) &= F'(x)d(\tilde{x}^j, x) \\ &= F'(x)e_j \\ &= \text{the } j\text{th column of } F'(x). \end{aligned}$$

This shows that $F(\tilde{x}^j) \notin V_{F(x)}$, which contradicts (a).

(b) $\Rightarrow$ (a). By Theorem 2.6, we have for $y \in V_x$,

$$d(F(y), F(x)) = F'(x)d(y, x),$$

so that $d(F(y), F(x)) \in V_{\mathbf{0}}$. Therefore $F(y) \in V_{F(x)}$, and hence $F(V_x) \subset V_{F(x)}$.

(c) $\Rightarrow$ (a). Trivial.

(a) $\Rightarrow$ (c). We prove by induction on m. By (a), the result is true for $m = 1$. Suppose (c) holds for $m = k$. We now show that $F^{k+1}(V_x) \subset V_{F^{k+1}(x)}$. By induction hypothesis and Theorem 2.6, we have for $y \in V_x$,

$$\begin{aligned} d(F^{k+1}(y), F^{k+1}(x)) &= F'(F^k(x))d(F^k(y), F^k(x)) \\ &= F'(F^k(x))e_j, \quad \text{for some } 1 \le j \le n. \end{aligned}$$

By the equivalence of (a) and (b), we have

$$d(F^{k+1}(y), F^{k+1}(x)) \in V_{\mathbf{0}},$$

so that $F^{k+1}(y) \in V_{F^{k+1}(x)}$. This completes the induction and the proof. ∎

The following result plays a crucial role in the proof of Theorem 3.1.

LEMMA 4.2. *Let* $F\colon \{0,1\}^n \to \{0,1\}^n$ *be with* $F(\xi) = \xi$. *Suppose*

(a) $\rho(F'(\xi)) = 0$,

(b) $F(V_x) \subset V_{F(x)}$ *for all* x *in* $\{0,1\}^n$.

Then ξ *is a global attractor for the network* (3.1).

Proof. Let

$$N_0 \stackrel{\text{def}}{=} \{\xi\}, \qquad N_1 \stackrel{\text{def}}{=} V_\xi \backslash \{\xi\},$$

$$N_i \stackrel{\text{def}}{=} \bigcup_{y \in N_{i-1}} (V_y \backslash \{y\}) \backslash N_{i-2}, \qquad (i = 2, \dots, n).$$

For $i = 0, 1, \dots, n$, let

$$M_i \stackrel{\text{def}}{=} \{x \in \{0,1\}^n;\ x \text{ has exactly } i \text{ components that differ from } \xi\text{s}\}.$$

Claim (i). "$N_i = M_i$ $(i = 0, 1, \dots, n)$."

We prove by induction on i $(0 \le i \le n)$. It is clear that $N_0 = M_0$. Suppose $N_k = M_k$ $(1 \le k < n)$. Let $x \in N_{k+1}$. Then there exists $y \in N_k$ such that

$$x \in (V_y \backslash \{y\}) \backslash N_{k-1}.$$

Because $y \in N_k$, x has exactly $k-1$ or $k+1$ components that differ from ξs. As $x \notin N_{k-1}$, we must have $x \in M_{k+1}$. This shows that $N_{k+1} \subset M_{k+1}$. To prove the reverse inclusion, let $x \in M_{k+1}$. Then there exists $y \in M_k = N_k$

such that $x \in V_y \backslash \{y\}$. Because $x \notin M_{k-1}$, by induction hypothesis, $x \notin N_{k-1}$, so that

$$x \in \bigcup_{y \in N_k} (V_y \backslash \{y\}) \backslash N_{k-1} = N_{k+1}.$$

Thus $M_{k+1} \subset N_{k+1}$, and hence $N_{k+1} = M_{k+1}$. This completes the inductive proof of Claim (i).

Claim (ii). "For each $i = 1, \ldots, n$, there exists a positive integer $p(\leq 2^n)$ such that $F^p(N_i) = N_0$."

The result is true for $i = 1$, because condition (a), Lemma 4.1, and Theorem 2.7 together imply that there is a positive integer $q \leq n$ such that $F^q(N_1) = N_0$. Assume that the assertion is true for $i = k < n$, that is, there exists a positive integer $s(\leq 2^n)$ such that

$$F^s(N_k) = N_0.$$

Let $x \in N_{k+1}$. Then there exists $y \in N_k$ such that $x \in V_y \backslash \{y\}$. Condition (b) and Lemma 4.1 together imply that

$$F^s(V_y) \subset V_{F^s(y)} = V_\xi,$$

so that $F^s(x) \in N_1 \cup \{\xi\}$. If $F^s(x) = \xi$, by taking $p = s$, then

$$F^p(x) \in N_0.$$

If $F^s(x) \in N_1$, by taking $p = q + s$, then

$$F^p(x) \in N_0.$$

It follows that $F^p(N_{k+1}) = N_0$. Evidently, $p \leq 2^n$ because $F(\xi) = \xi$. This completes the inductive proof of Claim (ii).

Because

$$\bigcup_{i=0}^{n} M_i = \{0, 1\}^n,$$

Claim (i) implies

$$\bigcup_{i=0}^{n} N_i = \{0, 1\}^n.$$

Hence, by Claim (ii), we conclude the proof of Lemma 4.2. ∎

LEMMA 4.3. *Let* F: $\{0,1\}^n \to \{0,1\}^n$. *Then the following conditions are equivalent*:

(a) $F(V_x) \subset V_{F(x)}$ $(x \in \{0,1\}^n)$,

(b) $\rho_{\mathrm{H}}(F(x), F(y)) \leq \rho_{\mathrm{H}}(x, y)$ $(x, y \in \{0,1\}^n)$.

Proof. As the implication (b) $\Rightarrow$ (a) is immediate, we need only to prove the implication (a) $\Rightarrow$ (b). Let $x, y \in \{0,1\}^n$ and $\rho_{\mathrm{H}}(x, y) = r$. Then $0 \leq r \leq n$. We may pass from x to y by a chain $[x = u_0, u_1, \ldots, u_{r-1}, u_r = y]$, which is denoted by $[x, y]$, where each point is a *neighbour* of the next point in the chain. Let us take the chain $[x, y]$ with *minimal length*. Then

$$\begin{aligned} d(F(x), F(y)) \leq{}& d(F(x), F(u_1)) + d(F(u_1), F(u_2)) \\ &+ \cdots + d(F(u_{r-1}), F(y)). \end{aligned}$$

Thus,

$$\begin{aligned} \rho_{\mathrm{H}}(F(x), F(y)) &= \sum_{i=1}^{n} \delta(f_i(x), f_i(y)) \\ &\leq \sum_{i=1}^{n} \delta(f_i(x), f_i(u_1)) + \sum_{i=1}^{n} \delta(f_i(u_1), f_i(u_2)) \\ &\quad + \cdots + \sum_{i=1}^{n} \delta(f_i(u_{r-1}), f_i(y)). \end{aligned}$$

By hypothesis (a), we obtain

$$\sum_{i=1}^{n} \delta(f_i(u_{j-1}), f_i(u_j)) \leq 1, \qquad (j = 1, \ldots, r),$$

so that

$$\rho_{\mathrm{H}}(F(x), F(y)) \leq r = \rho_{\mathrm{H}}(x, y).$$

This completes the proof. ∎

The following technical lemma concerning the *rearrangement* plays a vital role in the construction of a fixed point of a given map on $\{0,1\}^n$. The lemma can be established by *forward* and *backward* induction.

For any $m = 1, \ldots, n-1$ and for any subset $\{i_1, \ldots, i_{m+1}\}$ of $\{1, \ldots, n\}$, let

$$N_0[i_1, \ldots, i_{m+1}] \stackrel{\text{def}}{=} \{y \in \{0,1\}^n;\ y_{i_{m+1}} = 0,\ y_j = 0\ (j \neq i_1, \ldots, i_m)\},$$

$$N_1[i_1, \ldots, i_{m+1}] \stackrel{\text{def}}{=} \{y \in \{0,1\}^n;\ y_{i_{m+1}} = 1,\ y_j = 0\ (j \neq i_1, \ldots, i_m)\}.$$

LEMMA 4.4. *Let* $F\colon \{0,1\}^n \longrightarrow \{0,1\}^n$ *possess the following two properties*:

(i) $\rho(F'(x)) = 0$ *for all* x *in* $\{0,1\}^n$, *and*

(ii) $F(V_x) \subset V_{F(x)}$ *for all* x *in* $\{0,1\}^n$.

Suppose there exists a subset $\{i_1, \ldots, i_{m+1}\}$ *of* $\{1, \ldots, n\}$ *such that*

(iii) *there does not exist an* $\alpha \in N_0[i_1, \ldots, i_{m+1}]$ *such that* $f_j(\alpha) = \alpha_j$ $(j = i_1, \ldots, i_m)$, *but*

(iv) *there exists a* $\beta \in N_1[i_1, \ldots, i_{m+1}]$ *such that* $f_j(\beta) = \beta_j$ $(j = i_1, \ldots, i_m)$.

Let $\gamma \stackrel{\text{def}}{=} \tilde{\beta}^{i_{m+1}}$. *Then there exists a permutation* π *of* $\{i_1, \ldots, i_m\}$ *with* $\pi(\{i_1, \ldots, i_m\}) = \{j_1, \ldots, j_m\}$ *such that if* Λ *is a nonempty subset of* $\{j_1, \ldots, j_{m-1}\}$, *then*

$$F(\tilde{\gamma}^{\Lambda}) = \widetilde{F(\beta)}^{j_1, \Lambda^*},$$

where $\Lambda^* \stackrel{\text{def}}{=} \{j_{k+1};\ j_k \in \Lambda\}$.

Proof. Let $P(k)$ $(1 \leq k \leq m-1)$ be the statement: There exists a subset $\{j_1, \ldots, j_{k+1}\}$ of $\{i_1, \ldots, i_m\}$ with $j_s \neq j_t$ $(s, t = 1, \ldots, k+1)$ such that if Λ is a nonempty subset of $\{j_1, \ldots, j_k\}$, then $F(\tilde{\gamma}^{\Lambda}) = \widetilde{F(\beta)}^{j_1, \Lambda^*}$.

Because $\rho_{\mathrm{H}}(\gamma, \beta) = 1$, condition (ii) and Lemma 4.3 together imply that

$$\rho_{\mathrm{H}}(F(\gamma), F(\beta)) \leq 1.$$

It follows that

$$\rho_{\mathrm{H}}(F(\gamma), F(\beta)) = 1. \tag{4.1}$$

To see (4.1), if $F(\gamma) = F(\beta)$, then by condition (iv) we obtain

$$\gamma \in N_0[i_1, \ldots, i_{m+1}], \quad \text{and} \quad f_j(\gamma) = \gamma_j \qquad (j = i_1, \ldots, i_m),$$

which contradicts condition (iii). This contradiction proves (4.1). According to (4.1), there is a $j_1 \in \{1, \ldots, n\}$ such that

$$F(\gamma) = \widetilde{F(\beta)}^{j_1}. \tag{4.2}$$

It follows that $j_1 \in \{i_1, \ldots, i_m\}$. Indeed if $j_1 \notin \{i_1, \ldots, i_m\}$, then

$$f_j(\gamma) = \gamma_j, \qquad (j = i_1, \ldots, i_m),$$

which contradicts condition (iii) because $\gamma \in N_0[i_1, \ldots, i_{m+1}]$. Again by Lemma 4.3, we obtain

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_1}), F(\gamma)) \leq 1.$$

It follows that

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_1}), F(\gamma)) = 1. \tag{4.3}$$

To see this, if $F(\tilde{\gamma}^{j_1}) = F(\gamma)$, then by (4.2),

$$F(\tilde{\gamma}^{j_1}) = \widetilde{F(\beta)}^{j_1}.$$

Hence by condition (iv) we have $f_j(\tilde{\gamma}^{j_1}) = \tilde{\gamma}_j^{j_1}$ $(j = i_1, \ldots, i_m)$, which contradicts condition (iii) because $\tilde{\gamma}^{j_1} \in N_0[i_1, \ldots, i_{m+1}]$. This contradiction shows (4.3). According to (4.2) and (4.3), there is a $j_2 \in \{1, \ldots, n\}$ such that

$$F(\tilde{\gamma}^{j_1}) = \widetilde{F(\gamma)}^{j_2} = \widetilde{F(\beta)}^{j_1, j_2}. \tag{4.4}$$

Then $j_2 \in \{i_1, \ldots, i_m\}$. Indeed if $j_2 \notin \{i_1, \ldots, i_m\}$, then by (4.4),

$$f_j(\tilde{\gamma}^{j_1}) = \tilde{\gamma}_j^{j_1}, \qquad (j = i_1, \ldots, i_m),$$

which contradicts condition (iii) because $\tilde{\gamma}^{j_1} \in N_0[i_1, \ldots, i_{m+1}]$. Further $j_2 \neq j_1$. Indeed if $j_2 = j_1$, then by (4.4) we have

$$F(\tilde{\gamma}^{j_1}) = \widetilde{F(\gamma)}^{j_1},$$

so that the (j_1, j_1)-entry of $F'(\gamma)$ is 1, which contradicts Theorem 2.2 because $\rho(F'(\gamma)) = 0$ by condition (ii). Thus we have shown that there exists $\{j_1, j_2\} \subset \{i_1, \ldots, i_m\}$ with $j_1 \neq j_2$ such that if $\Lambda = \{j_1\}$, then

$$F(\tilde{\gamma}^{\Lambda}) = \widetilde{F(\beta)}^{j_1, \Lambda^*},$$

where $\Lambda^* = \{j_2\}$. Hence $P(1)$ is true.

We suppose that $P(k)$ is true for $k = 1, \ldots, m-2$. We now show that $P(m-1)$ is true. First, by Lemma 4.3 we have

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}), F(\tilde{\gamma}^{j_1,\ldots,j_{m-2}})) \leq 1.$$

It follows that

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}), F(\tilde{\gamma}^{j_1,\ldots,j_{m-2}})) = 1. \tag{4.5}$$

To see this, if $F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}) = F(\tilde{\gamma}^{j_1,\ldots,j_{m-2}})$, then by induction hypothesis,

$$F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}) = \widetilde{F(\beta)}^{j_1,\ldots,j_{m-1}}. \tag{4.6}$$

By (4.6) and condition (iv),

$$f_j(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}) = \tilde{\gamma}_j^{j_1,\ldots,j_{m-1}}, \qquad (j = i_1, \ldots, i_m),$$

which contradicts condition (iii) because $\tilde{\gamma}^{j_1,\ldots,j_{m-1}} \in N_0[i_1, \ldots, i_{m+1}]$. This contradiction proves (4.5). According to (4.5), there is a $j_m \in \{1, \ldots, n\}$ such that

$$F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}) = F(\tilde{\gamma}^{j_1,\ldots,j_{m-2}})^{\sim\, j_m},$$

so that by induction hypothesis,

$$F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}}) = \widetilde{F(\beta)}^{j_1,\ldots,j_m}. \tag{4.7}$$

For the sake of simplicity we use the notation $F(\tilde{\gamma}^{j_1,\dots,j_m})^{\sim\, j_m}$ to denote the j_mth neighbour of $F(\tilde{\gamma}^{j_1,\dots,j_m})$. We now show that $j_m \in \{i_1, \dots, i_m\}$. To see this, if $j_m \notin \{i_1, \dots, i_m\}$, then by (4.7),

$$f_j(\tilde{\gamma}^{j_1,\dots,j_{m-1}}) = \tilde{\gamma}_j^{j_1,\dots,j_{m-1}}, \qquad (j = i_1, \dots, i_m),$$

which contradicts condition (iii) because $\tilde{\gamma}^{j_1,\dots,j_{m-1}} \in N_0[i_1, \dots, i_{m+1}]$. Further $j_m \neq j_1, \dots, j_{m-1}$. Indeed, suppose $j_m \in \{j_1, \dots, j_{m-1}\}$. By induction hypothesis, we obtain

$$\begin{aligned}
F(\tilde{\gamma}^{j_1,\dots,j_{m-2}}) &= \widetilde{F(\beta)}^{j_1,\dots,j_{m-1}},\\
F(\tilde{\gamma}^{j_1,\dots,j_{m-3}}) &= \widetilde{F(\beta)}^{j_1,\dots,j_{m-2}},\\
F(\tilde{\gamma}^{j_1,\dots,j_{m-4},\, j_{m-2}}) &= \widetilde{F(\beta)}^{j_1,\dots,j_{m-3},\, j_{m-1}},\\
&\ \ \vdots\\
F(\tilde{\gamma}^{j_1,\, j_3,\dots,j_{m-2}}) &= \widetilde{F(\beta)}^{j_1,\, j_2,\, j_4,\dots,j_{m-1}},\\
F(\tilde{\gamma}^{j_2,\dots,j_{m-2}}) &= \widetilde{F(\beta)}^{j_1,\, j_3,\dots,j_{m-1}}.
\end{aligned}$$

We thus obtain the following principal submatrix of $F'(\tilde{\gamma}^{j_1,\dots,j_{m-2}})$ which has no zero columns,

$$\begin{array}{c} \\ j_1 \\ j_2 \\ \\ \\ j_m \\ \\ j_{m-2} \\ j_{m-1} \end{array}
\begin{array}{c}
\begin{array}{ccccc} j_1 & j_2 & & j_{m-2} & j_{m-1} \end{array}\\
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0\\
0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots\\
 & & & & 1\\
 & & & & \vdots\\
0 & 0 & & 0 & 0\\
0 & 0 & & 1 & 0
\end{pmatrix}
\end{array}.$$

By Theorem 2.1, we conclude that $\rho(F'(\tilde{\gamma}^{j_1,\dots,j_{m-2}})) = 1$, which contradicts condition (i). Thus we have shown that there exists a permutation π of $\{i_1, \dots, i_m\}$ with $\pi(\{i_1, \dots, i_m\}) = \{j_1, \dots, j_m\}$ such that

$$F(\tilde{\gamma}^{j_1,\dots,j_{m-1}}) = \widetilde{F(\beta)}^{j_1,\dots,j_m}. \tag{4.8}$$

For $\ell = 1, \dots, m-1$, let Σ^ℓ stand for the set of all indexed sets consisting of ℓ indices from $\{j_1, \dots, j_{m-1}\}$. To establish that $P(m-1)$ is true, we have to prove that for any $1 \le \ell \le m-1$ and for any $\Lambda_\ell \in \Sigma^\ell$,

$$F(\tilde{\gamma}^{\Lambda_\ell}) = \widetilde{F(\beta)}^{j_1,\, \Lambda_\ell^*}, \tag{4.9}$$

where $\Lambda_\ell^* \stackrel{\text{def}}{=} \{j_{k+1}; j_k \in \Lambda_\ell\}$. Let us now use *backward* induction. Because of (4.8), (4.9) is true for $\ell = m-1$. We suppose that (4.9) is true for $\ell = m-1, m-2, \dots, 2$. We now prove that (4.9) is true for $\ell = 1$. By induction hypothesis (i.e., $P(k)$ is true for $k = 2, \dots, m-2$), it is merely necessary to show that (4.9) is true for the indexed set $\Lambda_1 = \{j_{m-1}\}$. Because by (4.1) and (4.3),

$$F(\gamma) = \widetilde{F(\beta)}^{j_1}, \quad F(\tilde{\gamma}^{j_1}) = \widetilde{F(\beta)}^{j_1, j_2},$$

and by induction hypothesis,

$$F(\tilde{\gamma}^{j_2}) = \widetilde{F(\beta)}^{j_1, j_3}, \dots, F(\tilde{\gamma}^{j_{m-2}}) = \widetilde{F(\beta)}^{j_1, j_{m-1}},$$

we thus obtain

$$f_j(\tilde{\gamma}^{j_{m-1}}) = f_j(\gamma), \qquad (j = j_1, \dots, j_{m-1}). \tag{4.10}$$

To see (4.10), assume to the contrary that $f_j(\tilde{\gamma}^{j_{m-1}}) \neq f_j(\gamma)$ for some $j \in \{j_1, \dots, j_{m-1}\}$. Then we can find the following principal submatrix of $F'(\gamma)$ which has no zero columns,

$$\begin{array}{c c}
 & \begin{array}{ccccc} j_1 & j_2 & & j_{m-2} & j_{m-1} \end{array} \\
\begin{array}{c} j_1 \\ j_2 \\ \\ \\ j \\ \\ j_{m-2} \\ j_{m-1} \end{array} &
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
 & & & & 1 \\
 & & & & \vdots \\
0 & 0 & & 0 & 0 \\
0 & 0 & & 1 & 0
\end{pmatrix}
\end{array}.$$

By Theorem 2.1, we conclude that $\rho(F'(\gamma)) = 1$, which contradicts condition (i). This contradiction establishes (4.10). By Lemma 4.3,

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_{m-1}}), F(\gamma)) \le 1.$$

It follows that

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_{m-1}}), F(\gamma)) = 1. \tag{4.11}$$

To see (4.11), if $F(\tilde{\gamma}^{j_{m-1}}) = F(\gamma)$, then

$$\begin{aligned}
&\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_{m-1}}), F(\tilde{\gamma}^{j_1,\dots,j_{m-1}})) \\
&\quad = \rho_{\mathrm{H}}(F(\gamma), F(\tilde{\gamma}^{j_1,\dots,j_{m-1}})) \\
&\quad = \rho_{\mathrm{H}}(\widetilde{F(\beta)}^{j_1}, \widetilde{F(\beta)}^{j_1,\dots,j_m}), \quad \text{[by (4.1) and (4.8)]} \\
&\quad = m-1 \\
&\quad > \rho_{\mathrm{H}}(\tilde{\gamma}^{j_{m-1}}, \tilde{\gamma}^{j_1,\dots,j_{m-1}}),
\end{aligned}$$

which contradicts Lemma 4.3. This proves (4.11). According to (4.11) and (4.1), there is a ν in $\{1, \ldots, n\}$ such that

$$F(\tilde{\gamma}^{j_{m-1}}) = \widetilde{F(\gamma)}^{\nu} = \widetilde{F(\beta)}^{j_1, \nu}. \tag{4.12}$$

With a similar argument as before, we see that $\nu \in \{j_1, \ldots, j_m\}$. Further $\nu = j_m$. First $\nu \neq j_1$. If $\nu \neq j_m$, then $\nu = j_s$ for some $1 < s \leq m - 1$. Hence,

$$\begin{aligned} \rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_{m-1}}), F(\tilde{\gamma}^{j_s, j_{m-1}})) &= \rho_{\mathrm{H}}(\widetilde{F(\beta)}^{j_1, j_s}, \widetilde{F(\beta)}^{j_1, j_{s+1}, j_m}), \\ &\qquad \text{[by (4.12) and induction hypothesis]} \\ &> \rho_{\mathrm{H}}(\tilde{\gamma}^{j_{m-1}}, \tilde{\gamma}^{j_s, j_{m-1}}), \end{aligned}$$

which contradicts Lemma 4.3. Thus we have shown that

$$F(\tilde{\gamma}^{\Lambda_1}) = \widetilde{F(\beta)}^{j_1, \Lambda_1^*}.$$

This completes the *backward* induction and the proof of (4.9).

This completes the inductive proof of Lemma 4.4. ∎

We now turn to the proof of Theorem 3.1.

According to Lemma 4.2, it is merely necessary to show that F has a fixed point. The existence of a fixed point of F follows from the following Claim (i). (The proof of Claim (i) is the most intricate part of the whole proof of Theorem 3.1.)

Claim (i). "For any $k = 1, \ldots, n-1$ and for any subset $\{i_1, \ldots, i_{k+1}\}$ of $\{1, \ldots, n\}$, there exist a unique point,

$$\alpha \in N_0[i_1, \ldots, i_{k+1}],$$

and a unique point,

$$\beta \in N_1[i_1, \ldots, i_{k+1}],$$

such that $f_j(\alpha) = \alpha_j$ $(j = i_1, \ldots, i_k)$ and $f_j(\beta) = \beta_j$ $(j = i_1, \ldots, i_k)$."

Here the notations $N_0[i_1, \ldots, i_{k+1}]$ and $N_1[i_1, \ldots, i_{k+1}]$ are defined before Lemma 4.4.

We prove Claim (i) by induction on k $(1 \leq k < n)$. For any subset $\{i_1, i_2\}$ of $\{1, \ldots, n\}$, we have

$$N_0[i_1, i_2] = \{\mathbf{0}, \tilde{\mathbf{0}}^{i_1}\}.$$

Because $\rho(F'(\mathbf{0})) = 0$ by condition (b), Theorem 2.2 ensures that the (i_1, i_1)-entry of the discrete derivative $F'(\mathbf{0})$ is $f_{i_1 i_1}(\mathbf{0}) = 0$, so that $f_{i_1}(\mathbf{0}) = f_{i_1}(\tilde{\mathbf{0}}^{i_1})$. Hence one and only one of the statements $f_{i_1}(\mathbf{0}) = \mathbf{0}_{i_1}$, $f_{i_1}(\tilde{\mathbf{0}}^{i_1}) =$

$\tilde{\mathbf{0}}_{i_1}^{i_1}$ is true. Thus there exists a unique $\alpha \in N_0[i_1, i_2]$ such that $f_{i_1}(\alpha) = \alpha_{i_1}$. On the other hand,

$$N_1[i_1, i_2] = \{\tilde{\mathbf{0}}^{i_2}, \tilde{\mathbf{0}}^{i_1, i_2}\}.$$

A similar argument shows that there exists a unique $\beta \in \{\tilde{\mathbf{0}}^{i_2}, \tilde{\mathbf{0}}^{i_1, i_2}\}$ such that $f_{i_1}(\beta) = \beta_{i_1}$. Therefore Claim (i) is true for $k = 1$.

We suppose that Claim (i) is valid for $k = 1, \dots, m-1$ ($m < n$). We now establish that Claim (i) is true for $k = m$. We argue by contradiction, so we assume the following:

ASSERTION (i). *"There exists a subset $\{i_1, \dots, i_{m+1}\}$ of $\{1, \dots, n\}$ such that there does not exist an α in $N_0[i_1, \dots, i_{m+1}]$ such that $f_j(\alpha) = \alpha_j$ ($j = i_1, \dots, i_m$), or there does not exist a β in $N_1[i_1, \dots, i_{m+1}]$ such that $f_j(\beta) = \beta_j$ ($j = i_1, \dots, i_m$)."*

[A passing remark: If there exist an α in $N_0[i_1, \dots, i_{m+1}]$ and if there exist a β in $N_1[i_1, \dots, i_{m+1}]$ such that $f_j(\alpha) = \alpha_j$ ($j = i_1, \dots, i_m$) and $f_j(\beta) = \beta_j$ ($j = i_1, \dots, i_m$), then α and β are uniquely determined, by induction hypothesis.]

The *disjunction* in Assertion (i) can be replaced by the *conjunction*. More precisely, we have

ASSERTION (ii). *"There exists a subset $\{i_1, \dots, i_{m+1}\}$ of $\{1, \dots, n\}$ such that there does not exist an α in $N_0[i_1, \dots, i_{m+1}]$ such that $f_j(\alpha) = \alpha_j$ ($j = i_1, \dots, i_m$), and there does not exist a β in $N_1[i_1, \dots, i_{m+1}]$ such that $f_j(\beta) = \beta_j$ ($j = i_1, \dots, i_m$)."*

To see this, suppose the disjunction in Assertion (i) cannot be replaced by the conjunction. Then, without loss of generality we can assume that there exists a subset $\{i_1, \dots, i_{m+1}\}$ of $\{1, \dots, n\}$ such that there does not exist an α in $N_0[i_1, \dots, i_{m+1}]$ such that $f_j(\alpha) = \alpha_j$ ($j = i_1, \dots, i_m$), but there exists a β in $N_1[i_1, \dots, i_{m+1}]$ such that $f_j(\beta) = \beta_j$ ($j = i_1, \dots, i_m$). Let

$$\gamma \overset{\text{def}}{=} \tilde{\beta}^{i_{m+1}}.$$

Then conditions (a) and (b), and Lemma 4.4 together imply that there exists a rearrangement $\{j_1, \dots, j_m\}$ of $\{i_1, \dots, i_m\}$ such that

$$F(\tilde{\gamma}^{j_1, \dots, j_{m-1}}) = \widetilde{F(\beta)}^{j_1, \dots, j_m}. \tag{4.13}$$

With a similar argument as in the proof of Lemma 4.4, we can conclude that

$$\rho_{\mathrm{H}}(F(\tilde{\gamma}^{j_1, \dots, j_m}),\ F(\tilde{\gamma}^{j_1, \dots, j_{m-1}})) = 1,$$

so that there is a j_{m+1} in $\{1, \ldots, n\}$ such that

$$F(\tilde{\gamma}^{j_1,\ldots,j_m}) = F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})^{\sim\, j_{m+1}}. \tag{4.14}$$

Let us recall that the notation $F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})^{\sim\, j_{m+1}}$ denotes the j_{m+1}th neighbour of $F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})$. By (4.13) and (4.14), we thus have

$$F(\tilde{\gamma}^{j_1,\ldots,j_m}) = \widetilde{F(\beta)}^{j_1,\ldots,j_{m+1}}. \tag{4.15}$$

It follows that $j_{m+1} \in \{j_1, \ldots, j_m\}$, otherwise by (4.15),

$$f_j(\tilde{\gamma}^{j_1,\ldots,j_m}) = \tilde{\gamma}_j^{j_1,\ldots,j_m}, \qquad (j = j_1, \ldots, j_m),$$

in contradiction to the hypothesis of Assertion (i) because

$$\tilde{\gamma}^{j_1,\ldots,j_m} \in N_0[i_1, \ldots, i_{m+1}].$$

Thus we have shown that

$$F(\tilde{\gamma}^{j_1,\ldots,j_m}) = F(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})^{\sim\, j_{m+1}}, \qquad (j_{m+1} \in \{j_1, \ldots, j_m\}). \tag{4.16}$$

By (4.16), we can obtain the following principal submatrix of $F'(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})$ which has no zero columns,

$$\begin{array}{c c}
 & \begin{array}{ccccc} j_1 & j_2 & & j_{m-1} & j_m \end{array} \\
\begin{array}{c} j_1 \\ \\ \\ \\ j_{m+1} \\ \\ j_{m-1} \\ j_m \end{array} &
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
 & & & & 1 \\
 & & & & \vdots \\
0 & 0 & & 0 & 0 \\
0 & 0 & & 1 & 0
\end{pmatrix}
\end{array}.$$

By Theorem 2.1, we conclude that $\rho(F'(\tilde{\gamma}^{j_1,\ldots,j_{m-1}})) = 1$, which contradicts condition (a). This contradiction establishes Assertion (ii).

By induction hypothesis, we can associate to any index ν from $\{1, \ldots, m\}$, a unique point,

$$\alpha(i_\nu) \in N_0[i_1, \ldots, i_{\nu-1}, i_{\nu+1}, \ldots, i_{m+1}],$$

and a unique point,

$$\beta(i_\nu) \in N_1[i_1, \ldots, i_{\nu-1}, i_{\nu+1}, \ldots, i_{m+1}],$$

such that

$$
\begin{aligned}
f_i(\alpha(i_\nu)) &= \alpha(i_\nu)_i, \qquad (i \in \{i_1, \ldots, i_m\}\backslash\{i_\nu\}), \\
f_i(\beta(i_\nu)) &= \beta(i_\nu)_i, \qquad (i \in \{i_1, \ldots, i_m\}\backslash\{i_\nu\}).
\end{aligned} \tag{4.17}
$$

According to Assertion (ii), (4.17) implies

$$
\begin{aligned}
f_{i_\nu}(\alpha(i_\nu)) &= \overline{\alpha(i_\nu)}_{i_\nu}, \qquad (\nu = 1, \ldots, m), \\
f_{i_\nu}(\beta(i_\nu)) &= \overline{\beta(i_\nu)}_{i_\nu}, \qquad (\nu = 1, \ldots, m).
\end{aligned} \tag{4.18}
$$

ASSERTION (iii). *"There exists a permutation π of $\{i_1, \ldots, i_m\}$ with $\pi(\{i_1, \ldots, i_m\}) = \{j_1, \ldots, j_m\}$ such that for any nonempty subset Λ of $\{j_1, \ldots, j_{m-1}\}$,*

$$F(\widetilde{\alpha(j_1)}^{\Lambda}) = F(\alpha(j_1))^{\sim \Lambda^*},$$

where $\Lambda^* \stackrel{\text{def}}{=} \{j_{k+1};\ j_k \in \Lambda\}$."

Here $F(\alpha(j_1))^{\sim \Lambda^*}$ stands for $F(\alpha(j_1))^{\sim s_1, \ldots, s_p}$ if $\Lambda^* = \{s_1, \ldots, s_p\}$. The proof of Assertion (iii) is similar to that of Lemma 4.4. Because the proof is delicate, we include here the detailed proof for accuracy.

Let $P(k)$ be the statement: There exists a subset $\{j_1, \ldots, j_{k+1}\}$ of $\{i_1, \ldots, i_m\}$ such that for any nonempty subset Λ of $\{j_1, \ldots, j_k\}$,

$$F(\widetilde{\alpha(j_1)}^{\Lambda}) = F(\alpha(j_1))^{\sim \Lambda^*},$$

where $\Lambda^* \stackrel{\text{def}}{=} \{j_{k+1}; j_k \in \Lambda\}$.

Let

$$j_1 \stackrel{\text{def}}{=} i_1, \qquad \alpha(j_1) \stackrel{\text{def}}{=} a, \quad \text{and} \quad F(\alpha(j_1)) \stackrel{\text{def}}{=} b. \tag{4.19}$$

Because $\rho_{\mathrm{H}}(\tilde{a}^{j_1}, a) = 1$, condition (b) and Lemma 4.3 together imply that

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_1}), F(a)) \leq 1.$$

It follows that

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_1}), F(a)) = 1. \tag{4.20}$$

To see (4.20), if $F(\tilde{a}^{j_1}) = F(a)$, then

$$\tilde{a}^{j_1} \in N_0[i_1, \ldots, i_{m+1}],$$

and

$$f_j(\tilde{a}^{j_1}) = \tilde{a}^{j_1}_j, \qquad (j = i_1, \ldots, i_m),$$

which contradicts (4.18). This contradiction proves (4.20). According to (4.20), there is an index $j_2 \in \{1, \dots, n\}$ such that

$$F(\tilde{a}^{j_1}) = \widetilde{F(a)}^{j_2} = \tilde{b}^{j_2}. \tag{4.21}$$

It follows that $j_2 \in \{i_1, \dots, i_m\}$. Indeed if $j_2 \notin \{i_1, \dots, i_m\}$, then

$$f_j(\tilde{a}^{j_1}) = \tilde{a}_j^{j_1}, \qquad (j = i_1, \dots, i_m),$$

which contradicts (4.18). Further $j_2 \neq j_1$. Indeed if $j_2 = j_1$, then $F(\tilde{a}^{j_1}) = \widetilde{F(a)}^{j_1}$. Thus the (j_1, j_1)-entry of $F'(a)$ is 1, and so $\rho(F'(a)) = 1$ by Theorem 2.2, which contradicts condition (b). Thus we have shown that there exists $\{j_1, j_2\} \subset \{i_1, \dots, i_m\}$ with $j_1 \neq j_2$ such that if $\Lambda = \{j_1\}$, then

$$F(\tilde{a}^{\Lambda}) = \tilde{b}^{\Lambda^*},$$

where $\Lambda^* = \{j_2\}$. Hence $P(1)$ is true.

We suppose that $P(k)$ is true for $k = 1, \dots, m-2$. We now show that $P(m-1)$ is true. First,

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_1,\dots,j_{m-1}}),\ F(\tilde{a}^{j_1,\dots,j_{m-2}})) \leq 1,$$

by Lemma 4.3. It follows that

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_1,\dots,j_{m-1}}), F(\tilde{a}^{j_1,\dots,j_{m-2}})) = 1. \tag{4.22}$$

To see this, if $F(\tilde{a}^{j_1,\dots,j_{m-1}}) = F(\tilde{a}^{j_1,\dots,j_{m-2}})$, then by induction hypothesis,

$$F(\tilde{a}^{j_1,\dots,j_{m-1}}) = \tilde{b}^{j_1,\dots,j_{m-1}}. \tag{4.23}$$

By (4.22) and (4.19), we have

$$\tilde{a}^{j_1,\dots,j_{m-1}} \in N_0[i_1, \dots, i_{m+1}],$$

and

$$f_j(\tilde{a}^{j_1,\dots,j_{m-1}}) = \tilde{a}_j^{j_1,\dots,j_{m-1}}, \qquad (j = i_1, \dots, i_m),$$

which contradicts (4.18). This contradiction proves (4.22). According to (4.22), there is a $j_m \in \{1, \dots, n\}$ such that

$$F(\tilde{a}^{j_1,\dots,j_{m-1}}) = F(\tilde{a}^{j_1,\dots,j_{m-2}})^{\sim\, j_m},$$

so that by induction hypothesis,

$$F(\tilde{a}^{j_1,\dots,j_{m-1}}) = \tilde{b}^{j_2,\dots,j_m}. \tag{4.24}$$

We now show that $j_m \in \{i_1, \dots, i_m\}$. To see this, if $j_m \notin \{i_1, \dots, i_m\}$, then

$$\tilde{a}^{j_1,\dots,j_{m-1}} \in N_0[i_1, \dots, i_{m+1}],$$

$$f_j(\tilde{a}^{j_1,\dots,j_{m-1}}) = \tilde{a}_j^{j_1,\dots,j_{m-1}} \quad (j = i_1, \dots, i_m),$$

which contradicts (4.19). Hence $j_m \in \{j_1, \ldots, j_m\}$. Further $j_m \neq j_1, \ldots, j_{m-1}$. Indeed, suppose $j_m \in \{j_1, \ldots, j_{m-1}\}$. By induction hypothesis, we have

$$\begin{aligned} F(\tilde{a}^{j_1,\ldots,j_{m-2}}) &= \tilde{b}^{j_2,\ldots,j_{m-1}}, \\ F(\tilde{a}^{j_1,\ldots,j_{m-3}}) &= \tilde{b}^{j_2,\ldots,j_{m-2}}, \\ F(\tilde{a}^{j_1,\ldots,j_{m-4},\, j_{m-2}}) &= \tilde{b}^{j_2,\ldots,j_{m-3},\, j_{m-1}}, \\ &\vdots \\ F(\tilde{a}^{j_1,j_3,\ldots,j_{m-2}}) &= \tilde{b}^{j_2,\, j_4,\ldots,j_{m-1}}, \\ F(\tilde{a}^{j_2,\ldots,j_{m-2}}) &= \tilde{b}^{j_3,\ldots,j_{m-1}}. \end{aligned}$$

We thus obtain the following principal submatrix of $F'(\tilde{a}^{j_1,\ldots,j_{m-2}})$ which has no zero columns,

$$\begin{array}{c} \\ j_1 \\ j_2 \\ \\ \\ j_m \\ \\ j_{m-2} \\ j_{m-1} \end{array} \begin{array}{c} \begin{array}{ccccc} j_1 & j_2 & & j_{m-2} & j_{m-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 \\ & & & & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{pmatrix} \end{array}.$$

By Theorem 2.1, we conclude that $\rho(F'(\tilde{a}^{j_1,\ldots,j_{m-2}})) = 1$, which contradicts condition (a). Thus we have shown that there exists a rearrangement $\{j_1, \ldots, j_m\}$ of $\{i_1, \ldots, i_m\}$ such that

$$F(\tilde{a}^{j_1,\ldots,j_{m-1}}) = \tilde{b}^{j_2,\ldots,j_m}. \tag{4.25}$$

For $\ell = 1, \ldots, m-1$, let Σ^ℓ stand for the set of all indexed sets consisting of ℓ indices from $\{j_1, \ldots, j_{m-1}\}$. To establish that $P(m-1)$ is true, we have to prove that *for any* $1 \leq \ell \leq m-1$ *and for any* $\Lambda_\ell \in \Sigma^\ell$,

$$F(\tilde{a}^{\Lambda_\ell}) = \tilde{b}^{\Lambda_\ell^*}, \tag{4.26}$$

where $\Lambda_\ell^* \stackrel{\text{def}}{=} \{j_{k+1}\colon j_k \in \Lambda_\ell\}$. Let us now use *backward* induction. We suppose that (4.26) is true for $\ell = m-1, m-2, \ldots, 2$. We now prove that (4.26) is true for $\ell = 1$. By induction hypothesis (i.e., $P(k)$ is true for

$k = 2, \ldots, m-2$), it is merely necessary to show that (4.26) is true for the indexed set $\Lambda_1 = \{j_{m-1}\}$. Because by (4.19) and induction hypothesis,

$$F(a) = b, \qquad F(\tilde{a}^{j_1}) = \tilde{b}^{j_2}, \ldots, F(\tilde{a}^{j_{m-2}}) = \tilde{b}^{j_{m-1}},$$

we can conclude that

$$f_j(\tilde{a}^{j_{m-1}}) = f_j(a), \qquad (j = j_1, \ldots, j_{m-1}). \tag{4.27}$$

To see (4.27), assume to the contrary that $f_j(\tilde{a}^{j_{m-1}}) \neq f_j(a)$ for some $j \in \{j_1, \ldots, j_{m-1}\}$. Then we can obtain the following principal submatrix of $F'(a)$ which has no zero columns,

$$\begin{array}{c} \\ j_1 \\ j_2 \\ \\ \\ j \\ \\ j_{m-2} \\ j_{m-1} \end{array} \begin{array}{c} \begin{array}{ccccc} j_1 & j_2 & & j_{m-2} & j_{m-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 \\ & & & & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{pmatrix} \end{array}.$$

By Theorem 2.1, we conclude that $\rho(F'(a)) = 1$, which contradicts condition (a). This contradiction establishes (4.27). By Lemma 4.3,

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_{m-1}}), F(a)) \leq 1.$$

By the same argument as in the proof of Lemma 4.4, we thus obtain

$$\rho_{\mathrm{H}}(F(\tilde{a}^{j_{m-1}}), F(a)) = 1. \tag{4.28}$$

According to (4.28) and (4.19), there is an index s in $\{1, \ldots, n\}$ such that

$$F(\tilde{a}^{j_{m-1}}) = \widetilde{F(a)}^{s} = \tilde{b}^{s}. \tag{4.29}$$

With a similar argument as before, $s \in \{j_1, \ldots, j_m\}$. Further $s = j_m$. First, $s \neq j_1$. Suppose $s \neq j_m$, then $s = j_p$ for some $p \in \{2, \ldots, m-1\}$. Therefore,

$$\begin{aligned} \rho_{\mathrm{H}}(F(\tilde{a}^{j_{m-1}}), F(\tilde{a}^{j_1,\ldots,j_{m-1}})) &= \rho_{\mathrm{H}}(\tilde{b}^{j_p}, \tilde{b}^{j_2,\ldots,j_{m-1}}), \\ &\qquad \text{[by (4.29) and induction hypothesis]} \\ &= m-3 \\ &> \rho_{\mathrm{H}}(\tilde{a}^{j_{m-1}}, \tilde{a}^{j_1,\ldots,j_{m-1}}), \end{aligned}$$

which contradicts Lemma 4.3. Thus we have shown that

$$F(\tilde{a}^{\Lambda_1}) = \tilde{b}^{j_m} = \tilde{b}^{\Lambda_1^*}.$$

This completes the *backward* induction and the inductive proof of Assertion (iii).

Let

$$\alpha(j_1) \stackrel{\text{def}}{=} a, \qquad F(\alpha(j_1)) \stackrel{\text{def}}{=} b.$$

By Lemma 4.3 we have

$$\rho_{\mathrm{H}}(F(\widetilde{a}^{j_1,\dots,j_m}), F(\widetilde{a}^{j_1,\dots,j_{m-1}})) \leq 1.$$

An analogous argument as in the proof of Lemma 4.4, we see that

$$\rho_{\mathrm{H}}(F(\widetilde{a}^{j_1,\dots,j_m}), F(\widetilde{a}^{j_1,\dots,j_{m-1}})) = 1.$$

Thus by Assertion (iii), we can find an index j_{m+1} in $\{1, \dots, n\}$ such that

$$F(\widetilde{a}^{j_1,\dots,j_m}) = F(\widetilde{a}^{j_1,\dots,j_{m-1}})^{\sim\, j_{m+1}} = \widetilde{b}^{j_2,\dots,j_m,\, j_{m+1}}.$$

Further $j_{m+1} \in \{j_1, \dots, j_m\}$. Assertion (iii) therefore implies that the following principal submatrix of $F'(\widetilde{a}^{j_1,\dots,j_{m-1}})$ has no zero columns,

$$\begin{array}{cc}
 & \begin{array}{ccccc} j_1 & j_2 & & j_{m-1} & j_m \end{array} \\
\begin{array}{c} j_1 \\ j_2 \\ \\ \\ j_{m+1} \\ \\ j_{m-1} \\ j_m \end{array} &
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
 & & & & 1 \\
 & & & & \vdots \\
0 & 0 & & 0 & 0 \\
0 & 0 & & 1 & 0
\end{pmatrix}
\end{array}.$$

By Theorem 4.1, we conclude that $\rho(F'(\widetilde{a}^{j_1,\dots,j_{m-1}})) = 1$, which contradicts condition (a). This contradiction completes the inductive proof of Claim (i).

Claim (ii). "F has a fixed point."

Assume to the contrary that F has no fixed points. According to Claim (i), we can associate to any index i from $\{1, \dots, n\}$, a unique point $\alpha(i)$ in $\{0, 1\}^n$ with $\alpha(i)_i = 0$ and a unique point $\beta(i)$ in $\{0, 1\}^n$ with $\beta(i)_i = 1$ such that

$$f_j(\alpha(i)) = \alpha(i)_j, \qquad (j = 1, \dots, n,\ j \neq i),$$

and

$$f_j(\beta(i)) = \beta(i)_j, \qquad (j = 1, \ldots, n, \ j \neq i).$$

Because F has no fixed points, we must have

$$f_i(\alpha(i)) = \overline{\alpha(i)}_i, \qquad (i = 1, \ldots, n),$$

and

$$f_i(\beta(i)) = \overline{\beta(i)}_i, \qquad (i = 1, \ldots, n). \tag{4.30}$$

Based on (4.30), condition (a), and a similar argument as in the proof of Claim (i), we can construct a rearrangement $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$ with $i_1 = 1$ and the initial point,

$$\alpha(i_1) \stackrel{\text{def}}{=} a, \quad \text{with } F(\alpha(i_1)) \stackrel{\text{def}}{=} b,$$

such that

$$\begin{cases} F(\tilde{a}^{i_1}) = \tilde{b}^{i_2}, \\ \quad\vdots \\ F(\tilde{a}^{i_{n-1}}) = \tilde{b}^{i_n}, \\ F(\tilde{a}^{i_n}) = \tilde{b}^{\nu}, \quad \text{for some } \nu \in \{i_1, \ldots, i_{n-1}\}. \end{cases} \tag{4.31}$$

According to (4.31), it follows without difficulty that there exists a permutation matrix P such that

$$P^t F'(a) P = \begin{array}{c} \\ i_1 \\ i_2 \\ \\ \\ \nu \\ \\ i_{n-1} \\ i_n \end{array} \begin{array}{c} \begin{array}{ccccc} i_1 & i_2 & & i_{n-1} & i_n \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 \\ & & & & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{pmatrix} \end{array}.$$

Because $P^t F'(a) P$ contains no zero columns, by Theorem 2.2, we conclude that $\rho(F'(a)) = 1$, which contradicts condition (a). This contradiction proves Claim (ii).

This concludes our proof of Theorem 3.1. ∎

5. PROOF OF THEOREM 3.2

In order to prove Theorem 3.2 we shall employ the following lemmas.

LEMMA 5.1. *Let* $F: \{0,1\}^n \to \{0,1\}^n$. *If* $\rho(F'(x)) = 0$ *for all* $x \in \{0,1\}^n$, *then for any* $i, j \in \{1, \ldots, n\}$ *and* $x \in \{0,1\}^n$,

$$f_{ij}(x) = f_{ij}(\tilde{x}^i) = f_{ij}(\tilde{x}^j).$$

Proof. Let $i, j \in \{1, \ldots, n\}$ and $x \in \{0,1\}^n$. We have the following chain of equivalences,

$$\begin{aligned} f_{ij}(x) = 0 \quad &\Longleftrightarrow \quad f_i(x) = f_i(\tilde{x}^j) \\ &\Longleftrightarrow \quad f_i(\tilde{x}^j) = f_i(\tilde{x}^{j,j}) \\ &\Longleftrightarrow \quad f_{ij}(\tilde{x}^j) = 0. \end{aligned}$$

Hence $f_{ij}(x) = f_{ij}(\tilde{x}^j)$, so that $f_{ij}(\tilde{x}^i) = f_{ij}(\tilde{x}^{i,j})$. To complete the proof it is now enough to prove that $f_{ij}(x) = f_{ij}(\tilde{x}^{i,j})$. To see this, because by assumption $\rho(F'(x)) = \rho(F'(\tilde{x}^j)) = 0$, Theorem 2.2 implies that

$$f_i(x) = f_i(\tilde{x}^i), \quad \text{and} \quad f_i(\tilde{x}^j) = f_i(\tilde{x}^{i,j}).$$

Thus,

$$\begin{aligned} f_{ij}(x) = 0 \quad &\Longleftrightarrow \quad f_i(x) = f_i(\tilde{x}^j) \\ &\Longleftrightarrow \quad f_i(\tilde{x}^i) = f_i(\tilde{x}^{i,j}) \\ &\Longleftrightarrow \quad f_{ij}(\tilde{x}^{i,j}) = 0. \end{aligned}$$

This completes the proof. ∎

LEMMA 5.2. *Let* $F: \{0,1\}^n \to \{0,1\}^n$. *If* $\rho(F'(x)) = 0$ *for all* $x \in \{0,1\}^n$, *then for any* k *in* $\{1, \ldots, n\}$ *and* $x \in \{0,1\}^n$, $F'(x)$ *and* $F'(\tilde{x}^k)$ *have the same* k*th column and* k*th row.*

Proof. Let $k \in \{1, \ldots, n\}$ and $x \in \{0,1\}^n$. By Lemma 5.1, we have

$$f_{ik}(x) = f_{ik}(\tilde{x}^k), \qquad (i = 1, \ldots, n),$$

and

$$f_{kj}(x) = f_{kj}(\tilde{x}^k), \qquad (j = 1, \ldots, n).$$

The first equality shows that $F'(x)$ and $F'(\tilde{x}^k)$ have the same kth column, and the second equality shows that $F'(x)$ and $F'(\tilde{x}^k)$ have the same kth row.

This completes the proof. ∎

Remark. Lemmas 5.1 and 5.2 reveal the collective effect. Indeed if $\rho(F'(x)) = 0$ for some $x \in \{0, 1\}^n$, then there may exist $i, j, k \in \{1, \ldots, n\}$ such that

$$f_{ij}(x) \neq f_{ij}(\tilde{x}^i),$$

and $F'(x)$ and $F'(\tilde{x}^k)$ do not have the same kth row. To see this, let $F\colon \{0, 1\}^2 \to \{0, 1\}^2$ be defined by

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} \overline{x}_1 + \overline{x}_2 \\ x_1 x_2 \end{pmatrix}, \qquad (x \in \{0, 1\}^2).$$

Then $\rho(F'(0, 0)) = 0$. But

$$f_{12}(0, 0) \neq f_{12}(\widetilde{(0, 0)}^1),$$

and hence the first row of $F'(0, 0)$ differs from the first row of $F'(\widetilde{(0, 0)}^1)$.

The following *reduction lemma* plays a crucial role in our proof of (a) of Theorem 3.2.

LEMMA 5.3. *Let $F\colon \{0, 1\}^n \to \{0, 1\}^n$, $n \geq 2$, with $\rho(F'(x)) = 0$ for all $x \in \{0, 1\}^n$. For $j = 1, \ldots, n$, let the maps $G_j\colon \{0, 1\}^{n-1} \to \{0, 1\}^{n-1}$ and $H_j\colon \{0, 1\}^{n-1} \to \{0, 1\}^{n-1}$ be defined by*

$$G_j(y) \stackrel{\text{def}}{=} (f_1(\mu_y), \ldots, f_{j-1}(\mu_y), f_{j+1}(\mu_y), \ldots, f_n(\mu_y)), \qquad (y \in \{0, 1\}^{n-1}),$$

$$H_j(y) \stackrel{\text{def}}{=} (f_1(\lambda_y), \ldots, f_{j-1}(\lambda_y), f_{j+1}(\lambda_y), \ldots, f_n(\lambda_y)), \qquad (y \in \{0, 1\}^{n-1}),$$

respectively, where

$$\mu_y \stackrel{\text{def}}{=} (y_1, \ldots, y_{j-1}, 0, y_j, y_{j+1}, \ldots, y_{n-1}),$$

$$\lambda_y \stackrel{\text{def}}{=} (y_1, \ldots, y_{j-1}, 1, y_j, y_{j+1}, \ldots, y_{n-1}).$$

Then

(a) $\rho(G_j'(y)) = 0$ $(j = 1, \ldots, n,\ y \in \{0, 1\}^{n-1})$,

(b) $\rho(H_j'(y)) = 0$ $(j = 1, \ldots, n,\ y \in \{0, 1\}^{n-1})$.

Proof. We shall prove (a); the proof of (b) is quite similar. Suppose, by contradiction, that there exist $j \in \{1, \ldots, n\}$ and $y \in \{0, 1\}^{n-1}$ such that $\rho(G_j'(y)) = 1$. Because $\rho(G_j'(y)) = 1$, Theorem 2.1 implies that there exists a principal submatrix S of $G_j'(y)$ which has no zero rows. Let

$$S \stackrel{\text{def}}{=} \begin{pmatrix} s_{i_1, i_1} & \cdots & s_{i_1, i_k} \\ \vdots & & \vdots \\ s_{i_k, i_1} & \cdots & s_{i_k, i_k} \end{pmatrix},$$

and write

$$G_j(y) \stackrel{\text{def}}{=} (g_1^j(y), \ldots, g_{n-1}^j(y)), \qquad (y \in \{0,1\}^{n-1}).$$

Because S has no zero rows, we can associate to any index i from $\{i_1, \ldots, i_k\}$, an index ℓ_i in $\{i_1, \ldots, i_k\}$ such that $s_{i,\ell_i} = 1$, that is,

$$g_i^j(y) \neq g_i^j(\tilde{y}^{\ell_i}).$$

For i in $\{i_1, \ldots, i_k\}$, define

$$\nu(i) \stackrel{\text{def}}{=} \begin{cases} i, & i \leq j-1, \\ i+1, & i \geq j. \end{cases}$$

Then for any index i from $\{i_1, \ldots, i_k\}$, we have

$$f_{\nu(i)}(\mu_y) = g_i^j(y) \neq g_i^j(\tilde{y}^{\ell_i}) = f_{\nu(i)}(\widetilde{\mu_y}^{\nu(\ell_i)}),$$

so that $f_{\nu(i)\nu(\ell_i)}(\mu_y) = 1$. Thus we can associate to any index $\nu(i)$ $(i = i_1, \ldots, i_k)$, an index $\nu(\ell_i)$ such that

$$f_{\nu(i)\nu(\ell_i)}(\mu_y) = 1.$$

Accordingly, $F'(\mu_y)$ contains a principal submatrix which has no zero rows, so that $\rho(F'(\mu_y)) = 1$ by Theorem 2.1, which contradicts the hypothesis. This proves (a).

This completes the proof. ∎

We now turn to the proof of (a) of Theorem 3.2. The case $n = 1$ holds trivially because the hypothesis $\rho(F'(0)) = \rho(F'(1)) = 0$ shows that $F(0) = F(1)$. Thus the assertion follows.

Suppose $n = 2$. Lemma 5.2 ensures that

$$F'(0,0) = F'(1,0) = F'(1,1) = F'(0,1).$$

Hence by the hypothesis and Theorem 2.5, $\rho(B(F)) = 0$, so that F is a contraction and the assertion follows from Theorem 2.4.

Suppose $n = 3$. We have observed that F is *not necessarily* a contraction in Section 3. For $j = 1, 2, 3$, define $G_j\colon \{0,1\}^2 \to \{0,1\}^2$ and $H_j\colon \{0,1\}^2 \to \{0,1\}^2$ by

$$G_1(y) \stackrel{\text{def}}{=} (f_2(0, y_1, y_2), f_3(0, y_1, y_2)), \qquad (y \in \{0,1\}^2),$$

$$G_2(y) \stackrel{\text{def}}{=} (f_1(y_1, 0, y_2), f_3(y_1, 0, y_2)), \qquad (y \in \{0,1\}^2),$$

$$G_3(y) \stackrel{\text{def}}{=} (f_1(y_1, y_2, 0), f_2(y_1, y_2, 0)), \qquad (y \in \{0,1\}^2),$$

$$H_1(y) \stackrel{\text{def}}{=} (f_2(1, y_1, y_2), f_3(1, y_1, y_2)), \qquad (y \in \{0,1\}^2),$$

$$H_2(y) \stackrel{\text{def}}{=} (f_1(y_1, 1, y_2), f_3(y_1, 1, y_2)), \qquad (y \in \{0,1\}^2),$$

$$H_3(y) \stackrel{\text{def}}{=} (f_1(y_1, y_2, 1), f_2(y_1, y_2, 1)), \qquad (y \in \{0,1\}^2),$$

respectively. For $i = 1, 2, 3$ and $y \in \{0, 1\}^2$, we write

$$G_i(y) \stackrel{\text{def}}{=} (g_1^i(y), g_2^i(y)),$$

$$H_i(y) \stackrel{\text{def}}{=} (h_1^i(y), h_2^i(y)).$$

Then for $j = 1, 2, 3$, Lemma 5.3 implies that $\rho(G_j'(y)) = 0$ for all $y \in \{0, 1\}^2$ and $\rho(H_j'(y)) = 0$ for all $y \in \{0, 1\}^2$. Thus by case $n = 2$, corresponding to each $j = 1, 2, 3$ there exist ξ^j and η^j in $\{0, 1\}^2$ such that

$$G_j(\xi^j) = \xi^j, \quad \text{and} \quad H_j(\eta^j) = \eta^j.$$

For $j, k = 1, 2, 3$, let

$$a_{j,k} \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } k = j, \\ \xi_k^j, & \text{if } k \leq j - 1, \\ \xi_{k-1}^j, & \text{if } k \geq j, \end{cases} \qquad b_{j,k} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k = j, \\ \eta_k^j, & \text{if } k \leq j - 1, \\ \eta_{k-1}^j, & \text{if } k \geq j. \end{cases}$$

Let

$$a_i \stackrel{\text{def}}{=} (a_{i,1}, a_{i,2}, a_{i,3}) \quad \text{and} \quad b_i \stackrel{\text{def}}{=} (b_{i,1}, b_{i,2}, b_{i,3}), \qquad (i = 1, 2, 3).$$

Then

$$\begin{aligned} F(a_1) &= F(a_{1,1}, a_{1,2}, a_{1,3}) \\ &= F(0, \xi_1^1, \xi_2^1) \\ &= (f_1(0, \xi_1^1, \xi_2^1), g_1^1(\xi_1^1, \xi_2^1), g_2^1(\xi_1^1, \xi_2^1)) \\ &= (f_1(0, \xi_1^1, \xi_2^1), \xi_1^1, \xi_2^1) \\ &= (f_1(0, \xi_1^1, \xi_2^1), a_{1,2}, a_{1,3}). \end{aligned}$$

The preceding argument shows also that

$$\begin{aligned} F(a_2) &= (a_{2,1}, f_2(\xi_1^2, 0, \xi_2^2), a_{2,3}), \\ F(a_3) &= (a_{3,1}, a_{3,2}, f_3(\xi_1^3, \xi_2^3, 0)), \\ F(b_1) &= (f_1(1, \eta_1^1, \eta_2^1), b_{1,2}, b_{1,3}), \\ F(b_2) &= (b_{2,1}, f_2(\eta_1^2, 1, \eta_2^2), b_{2,3}), \\ F(b_3) &= (b_{3,1}, b_{3,2}, f_3(\eta_1^3, \eta_2^3, 1)). \end{aligned}$$

Claim (i). "There exists an element $\alpha \in \{a_1, a_2, a_3, b_1, b_2, b_3\}$ such that $F(\alpha) = \alpha$."

TABLE 7

Bit string x	Bit string $F(x)$
$0\, a_{1,2}\, a_{1,3}$	$1\, a_{1,2}\, a_{1,3}$
$a_{2,1}\, 0\, a_{2,3}$	$a_{2,1}\, 1\, a_{2,3}$
$a_{3,1}\, a_{3,2}\, 0$	$a_{3,1}\, a_{3,2}\, 1$
$1\, b_{1,2}\, b_{1,3}$	$0\, b_{1,2}\, b_{1,3}$
$b_{2,1}\, 1\, b_{2,3}$	$b_{2,1}\, 0\, b_{2,3}$
$b_{3,1}\, b_{3,2}\, 1$	$b_{3,1}\, b_{3,2}\, 0$

Suppose, by contradiction, that $F(x) \neq x$ for all $x \in \{a_1, a_2, a_3, b_1, b_2, b_3\}$. Then we have

$$F(a_1) = \widetilde{a_1}^1, \qquad F(a_2) = \widetilde{a_2}^2, \qquad F(a_3) = \widetilde{a_3}^3,$$

$$F(b_1) = \widetilde{b_1}^1, \qquad F(b_2) = \widetilde{b_2}^2, \qquad F(b_3) = \widetilde{b_3}^3.$$

Hence we have Table 7.

OBSERVATION. "*$a_{i,j} = \overline{a_{j,i}}$ $(i, j = 1, 2, 3,\ i \neq j)$ and $b_{i,j} = \overline{a_{i,j}}$ $(i, j = 1, 2, 3)$.*"

First we observe that the six elements $a_1, a_2, a_3, b_1, b_2, b_3$ are pairwise distinct, that is, $a_i \neq a_j$, $b_i \neq b_j$, and $a_i \neq b_j$ $(i, j = 1, 2, 3)$. To see this, if $a_i = a_j$ for some $i \neq j$, then $a_{i,j} = a_{j,j} = 0$. Hence $f_j(a_i) = a_{i,j} = 0$, in contradiction to $f_j(a_j) = 1$. The proofs of other cases are analogous.

Furthermore $a_{i,j} = \overline{a_{j,i}}$ for all $i, j = 1, 2, 3$, with $i \neq j$, and $b_{i,j} = \overline{a_{i,j}}$ for all $i, j = 1, 2, 3$. To see this, if $a_{i,j} = 0$ for $i \neq j$, then $a_{j,i} = 1$ (otherwise by the previous observation $a_j = \widetilde{a_i}^k$ for $k \neq i, j$, but Theorem 2.2 gives $f_{kk}(a_i) = 0$, and so $a_{i,k} = (\widetilde{a_i}^k)_k = a_{j,k}$, contradicting $a_{j,k} = \overline{a_{i,k}}$). Hence we must have $b_{i,j} = 1$ (otherwise $b_{i,j} = 0$ which implies $\widetilde{b_i}^k = a_j$ for $k \neq i, j$ by the foregoing observation. Because Theorem 2.2 gives $f_{kk}(b_i) = 0$, i.e., $b_{i,k} = (\widetilde{b_i}^k)_k = a_{j,k}$, a contradiction), and so $b_{j,i} = 0$ (otherwise $b_{j,i} = 1$ by the preceding observation, so $b_i = \widetilde{b_j}^k$ for $k \neq i, j$. Hence $b_{j,k} = (\widetilde{b_j}^k)_k$, because $f_{kk}(b_j) = 0$ by Theorem 2.2. This implies $b_{j,k} = b_{i,k}$, in contradiction.) Therefore,

$$a_{i,j} = 0 \quad \Rightarrow \quad a_{j,i} = 1 \quad \Rightarrow \quad b_{i,j} = 1 \quad \Rightarrow \quad b_{j,i} = 0,$$

and we have the desired conclusion.

The previous observation now makes it clear that Table 7 becomes Table 8.

Case 1. $(a_{1,2}, a_{1,3}, a_{2,3}) = (0, 0, 0)$. By Table 8, we have Table 9. Since by hypothesis $\rho(F'(x)) = 0$ for all $x \in \{0, 1\}^3$, Theorem 2.2 implies

TABLE 8

Bit string x	Bit string $F(x)$
$0\,a_{1,2}\,a_{1,3}$	$1\,a_{1,2}\,a_{1,3}$
$\overline{a}_{1,2}\,0\,a_{2,3}$	$\overline{a}_{1,2}\,1\,a_{2,3}$
$\overline{a}_{1,3}\,\overline{a}_{2,3}\,0$	$\overline{a}_{1,3}\,\overline{a}_{2,3}\,1$
$1\,\overline{a}_{1,2}\,\overline{a}_{1,3}$	$0\,\overline{a}_{1,2}\,\overline{a}_{1,3}$
$a_{1,2}\,1\,\overline{a}_{2,3}$	$a_{1,2}\,0\,\overline{a}_{2,3}$
$a_{1,3}\,a_{2,3}\,1$	$a_{1,3}\,a_{2,3}\,0$

TABLE 9

Bit string x	000	100	110	111	011	001
Bit string $F(x)$	100	110	111	011	001	000

that $f_{11}(1,1,0) = f_{22}(0,0,0) = f_{33}(0,1,1) = 0$, so that $f_1(0,1,0) = f_3(0,1,0) = 1$ and $f_2(0,1,0) = 0$. Hence,

$$F'(0,0,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,0)) = 1$, which contradicts the hypothesis.

Case 2. $(a_{1,2}, a_{1,3}, a_{2,3}) = (0,0,1)$. By Table 8, we have Table 10. By Theorem 2.2, $f_{11}(1,0,1) = f_{22}(0,1,1) = f_{33}(0,0,0) = 0$, so that $f_1(0,0,1) = f_2(0,0,1) = 1$ and $f_3(0,0,1) = 0$. Hence,

$$F'(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,0)) = 1$, which contradicts the hypothesis.

Case 3. $(a_{1,2}, a_{1,3}, a_{2,3}) = (0,1,0)$. By Table 8, we have Table 11. By Theorem 2.2, $f_{11}(1,0,0) = f_{22}(0,1,0) = f_{33}(0,0,1) = 0$, so that

TABLE 10

Bit string x	000	101	100	111	010	011
Bit string $F(x)$	100	111	101	011	000	010

TABLE 11

Bit string x	001	100	010	110	011	101
Bit string $F(x)$	101	110	011	010	001	100

TABLE 12

Bit string x	001	101	000	110	010	111
Bit string $F(x)$	101	111	001	010	000	110

$f_1(0,0,0) = f_2(0,0,0) = f_3(0,0,0) = 1$. Hence,

$$F'(0,0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,1)) = 1$, which contradicts the hypothesis.

Case 4. $(a_{1,2}, a_{1,3}, a_{2,3}) = (0,1,1)$. By Table 8, we have Table 12. By Theorem 2.2, $f_{11}(1,1,1) = f_{22}(0,0,1) = f_{33}(0,1,0) = 0$, so that $f_1(0,1,1) = 1$ and $f_2(0,1,1) = f_3(0,1,1) = 0$. Hence,

$$F'(0,0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,1)) = 1$, which contradicts the hypothesis.

Case 5. $(a_{1,2}, a_{1,3}, a_{2,3}) = (1,0,0)$. By Table 8, we have Table 13. By Theorem 2.2, $f_{11}(1,1,1) = f_{22}(0,0,1) = f_{33}(0,1,0) = 0$, so that $f_1(0,1,1) = 1$ and $f_2(0,1,1) = f_3(0,1,1) = 0$. Hence,

$$F'(0,0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,1)) = 1$, which contradicts the hypothesis.

Case 6. $(a_{1,2}, a_{1,3}, a_{2,3}) = (1,0,1)$. By Table 8, we have Table 14. By Theorem 2.2, $f_{11}(1,0,0) = f_{22}(0,1,0) = f_{33}(0,0,1) = 0$, so that

TABLE 13

Bit string x	010	000	110	101	111	001
Bit string $F(x)$	110	010	111	001	101	000

TABLE 14

Bit string x	010	001	100	101	110	011
Bit string $F(x)$	110	011	101	001	100	010

$f_1(0,0,0) = f_2(0,0,0) = f_3(0,0,0) = 1$. Hence,

$$F'(0,0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,1)) = 1$, which contradicts the hypothesis.

Case 7. $(a_{1,2}, a_{1,3}, a_{2,3}) = (1,1,0)$. By Table 8, we have Table 15. By Theorem 2.2, $f_{11}(1,0,1) = f_{22}(0,1,1) = f_{33}(0,0,0) = 0$, so that $f_1(0,0,1) = 1 = f_2(0,0,1)$ and $f_3(0,0,1) = 0$. Hence,

$$F'(0,0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,0,1)) = 1$, which contradicts the hypothesis.

Case 8. $(a_{1,2}, a_{1,3}, a_{2,3}) = (1,1,1)$. By Table 8, we have Table 16. By Theorem 2.2, $f_{11}(1,1,0) = f_{22}(0,0,0) = f_{33}(0,1,1) = 0$, so that $f_1(0,1,0) = f_3(0,1,0) = 1$ and $f_2(0,1,0) = 0$. Hence,

$$F'(0,1,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.1, $\rho(F'(0,1,1)) = 1$, which contradicts the hypothesis.

Thus we arrive at a contradiction for all Cases 1–8. This contradiction completes the proof of Claim (i).

Claim (ii). "If $F(\xi) = \xi$, then ξ is a global attractor for the network (3.1)."

Because $\rho(F'(\xi)) = 0$, Theorem 2.3 implies that there is a permutation matrix P such that $P^t F'(\xi) P$ is strictly upper triangular. Let

$$P^t F'(\xi) P = \begin{pmatrix} 0 & f_{ij}(\xi) & f_{ik}(\xi) \\ 0 & 0 & f_{jk}(\xi) \\ 0 & 0 & 0 \end{pmatrix},$$

TABLE 15

Bit string x	011	000	010	100	111	101
Bit string $F(x)$	111	010	011	000	101	100

TABLE 16

Bit string x	011	001	000	100	110	111
Bit string $F(x)$	111	011	001	000	100	110

TABLE 17

x	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^j$	$\tilde{\xi}^k$	$\tilde{\xi}^{i,j}$	$\tilde{\xi}^{i,k}$	$\tilde{\xi}^{j,k}$
$F(x)$	ξ	ξ	ξ	ξ	ξ or $\tilde{\xi}^k$	ξ or $\tilde{\xi}^j$	ξ or $\tilde{\xi}^i$
$F^2(x)$	ξ	ξ	ξ	ξ	ξ	ξ	ξ

where $\{i, j, k\}$ is a rearrangement of $\{1, 2, 3\}$. We divide the proof into eight cases.

Case 1. $f_{ij}(\xi) = f_{ik}(\xi) = f_{jk}(\xi) = 0$. Then $F'(\xi)$ is the zero matrix. Thus we have Table 17. Because $\rho(F'(\tilde{\xi}^{i,j,k})) = 0$, by Theorem 2.2 $F'(\tilde{\xi}^{i,j,k})$ has a zero column, so that there exists $a \in V_{\tilde{\xi}^{i,j,k}}$ with $a \neq \tilde{\xi}^{i,j,k}$ such that $F(a) = F(\tilde{\xi}^{i,j,k})$. Hence $\tilde{\xi}^{i,j,k}$ is not a fixed point of F, and so

$$F(\tilde{\xi}^{i,j,k}) \in \{\xi, \tilde{\xi}^i, \tilde{\xi}^j, \tilde{\xi}^k, \tilde{\xi}^{i,j}, \tilde{\xi}^{i,k}, \tilde{\xi}^{j,k}\}.$$

It follows from Table 17 that $F^3 \equiv \xi$.

Case 2. $f_{jk}(\xi) = 1$ and $f_{ij}(\xi) = f_{ik}(\xi) = 0$. Then we have Table 18. A similar argument as in Case 1 now shows that

$$F(\tilde{\xi}^{i,j,k}) \in \{\xi, \tilde{\xi}^i, \tilde{\xi}^j, \tilde{\xi}^k, \tilde{\xi}^{i,j}, \tilde{\xi}^{i,k}, \tilde{\xi}^{j,k}\}.$$

It follows from Table 18 that $F^5 \equiv \xi$.

Case 3. $f_{ij}(\xi) = 1$, and $f_{ik}(\xi) = f_{jk}(\xi) = 0$. The same argument as in Case 2 shows that $F^5 \equiv \xi$.

Case 4. $f_{ik}(\xi) = 1$, and $f_{ij}(\xi) = f_{jk}(\xi) = 0$. The same argument as in Case 2 shows that $F^5 \equiv \xi$.

Case 5. $f_{ij}(\xi) = f_{ik}(\xi) = 1$, and $f_{jk}(\xi) = 0$. Then we have Table 19.

If $F(\tilde{\xi}^{i,k}) = \tilde{\xi}^{i,j}$ and $F(\tilde{\xi}^{i,j}) = \tilde{\xi}^{i,k}$ simultaneously, we see that $F'(\tilde{\xi}^i)$ contains a principal submatrix which has no zero rows. Hence $\rho(F'(\tilde{\xi}^i)) = 1$ by Theorem 2.1, which contradicts the hypothesis. Thus,

$$F(\tilde{\xi}^{i,k}) \neq \tilde{\xi}^{i,j} \quad \text{or} \quad F(\tilde{\xi}^{i,j}) \neq \tilde{\xi}^{i,k}.$$

TABLE 18

x	$F(x)$	$F^2(x)$	$F^3(x)$	$F^4(x)$
ξ	ξ	ξ	ξ	ξ
$\tilde{\xi}^i$	ξ	ξ	ξ	ξ
$\tilde{\xi}^j$	ξ	ξ	ξ	ξ
$\tilde{\xi}^k$	$\tilde{\xi}^j$	ξ	ξ	ξ
$\tilde{\xi}^{i,j}$	ξ or $\tilde{\xi}^k$	ξ or $\tilde{\xi}^j$	ξ	ξ
$\tilde{\xi}^{i,k}$	ξ or $\tilde{\xi}^j$	ξ	ξ	ξ
$\tilde{\xi}^{j,k}$	$\tilde{\xi}^j$ or $\tilde{\xi}^{i,j}$	ξ or $\tilde{\xi}^k$	ξ or $\tilde{\xi}^j$	ξ

TABLE 19

x	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^j$	$\tilde{\xi}^k$	$\tilde{\xi}^{i,j}$	$\tilde{\xi}^{i,k}$	$\tilde{\xi}^{j,k}$
$F(x)$	ξ	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^i$	$\tilde{\xi}^i$ or $\tilde{\xi}^{i,k}$	$\tilde{\xi}^i$ or $\tilde{\xi}^{i,j}$	ξ or $\tilde{\xi}^i$

A similar argument as in Case 1 now shows that $\tilde{\xi}^{i,j,k}$ is not a fixed point of F, so that

$$F(\tilde{\xi}^{i,j,k}) \in \{\xi, \tilde{\xi}^i, \tilde{\xi}^j, \tilde{\xi}^k, \tilde{\xi}^{i,j}, \tilde{\xi}^{i,k}, \tilde{\xi}^{j,k}\}.$$

It follows from Table 19 that $F^4 \equiv \xi$.

Case 6. $f_{ij}(\xi) = f_{jk}(\xi) = 1$, and $f_{ik}(\xi) = 0$. The same argument as in Case 1 shows that $F^6 \equiv \xi$.

Case 7. $f_{ik}(\xi) = f_{jk}(\xi) = 1$, and $f_{ij}(\xi) = 0$. Then we have Table 20.

If $F(\tilde{\xi}^k) = \tilde{\xi}^{i,j}$ and $F(\tilde{\xi}^{i,j}) = \tilde{\xi}^k$ simultaneously, we see that $F'(\tilde{\xi}^{i,j,k})$ contains a principal submatrix which has no zero rows. Hence $\rho(F'(\tilde{\xi}^{i,j,k})) = 1$ by Theorem 2.1, which contradicts the hypothesis. Thus,

$$F(\tilde{\xi}^k) \neq \tilde{\xi}^{i,j} \quad \text{or} \quad F(\tilde{\xi}^{i,j}) \neq \tilde{\xi}^k.$$

A similar argument as in Case 1 now shows that $\tilde{\xi}^{i,j,k}$ is not a fixed point of F, so that

$$F(\tilde{\xi}^{i,j,k}) \in \{\xi, \tilde{\xi}^i, \tilde{\xi}^j, \tilde{\xi}^k, \tilde{\xi}^{i,j}, \tilde{\xi}^{i,k}, \tilde{\xi}^{j,k}\}.$$

It follows from Table 20 that $F^3 \equiv \xi$.

Case 8. $f_{ij}(\xi) = f_{ik}(\xi) = f_{jk}(\xi) = 1$. Then we have Table 21.

A similar argument as in Case 5 now shows that

$$F(\tilde{\xi}^{i,k}) \neq \tilde{\xi}^{i,j}, \quad \text{or} \quad F(\tilde{\xi}^{i,j}) \neq \tilde{\xi}^{i,k}.$$

Because $\tilde{\xi}^{i,j,k}$ is not a fixed point of F, it follows that

$$F(\tilde{\xi}^{i,j,k}) \in \{\xi, \tilde{\xi}^i, \tilde{\xi}^j, \tilde{\xi}^k, \tilde{\xi}^{i,j}, \tilde{\xi}^{i,k}, \tilde{\xi}^{j,k}\}.$$

TABLE 20

x	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^j$	$\tilde{\xi}^k$	$\tilde{\xi}^{i,j}$	$\tilde{\xi}^{i,k}$	$\tilde{\xi}^{j,k}$
$F(x)$	ξ	ξ	ξ	$\tilde{\xi}^{i,j}$	ξ or $\tilde{\xi}^k$	$\tilde{\xi}^i$ or $\tilde{\xi}^{i,j}$	$\tilde{\xi}^j$ or $\tilde{\xi}^{i,j}$

TABLE 21

x	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^j$	$\tilde{\xi}^k$	$\tilde{\xi}^{i,j}$	$\tilde{\xi}^{i,k}$	$\tilde{\xi}^{j,k}$
$F(x)$	ξ	ξ	$\tilde{\xi}^i$	$\tilde{\xi}^{i,j}$	$\tilde{\xi}^i$ or $\tilde{\xi}^{i,k}$	$\tilde{\xi}^i$ or $\tilde{\xi}^{i,j}$	$\tilde{\xi}^j$ or $\tilde{\xi}^{i,j}$

TABLE 22

Bit string x	0000	1000	0100	0010	0001	1100	1010	1001
Bit string $F(x)$	0000	0000	0000	1100	1010	0001	1100	1000

TABLE 23

Bit string x	0110	0101	0011	1110	1101	1011	0111	1111
Bit string $F(x)$	0100	0000	1010	0100	0001	1000	0000	0000

It follows from Table 21 that $F^4 \equiv \xi$. This completes the proof of Claim (ii). This concludes our proof of (a) of Theorem 3.2.

(b) Let F: $\{0,1\}^4 \to \{0,1\}^4$ be defined by

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} \overline{x}_2 x_3 + \overline{x}_2 x_4 \\ x_3\, \overline{x}_4 \\ \overline{x}_1\, \overline{x}_2\, x_4 \\ x_1\, x_2\, \overline{x}_3 \end{pmatrix}, \qquad (x \in \{0,1\}^4).$$

Then F is given by Tables 22 and 23. The discrete derivatives of F are the following,

$$F'(0,0,0,0) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(1,0,0,0) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$F'(0,1,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(0,0,1,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(0,0,0,1) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(1,1,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$F'(1,0,1,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(1,0,0,1) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$F'(0,1,1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(0,1,0,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(0,0,1,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(1,1,1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$F'(1,1,0,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad F'(1,0,1,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(0,1,1,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F'(1,1,1,1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, it is readily seen that $\rho(F'(x)) = 0$ for all x in $\{0,1\}^4$. But the unique fixed point $\mathbf{0}$ of F is not a global attractor because

$$\{(0,0,0,1),\ (1,0,1,0),\ (1,1,0,0)\}$$

is a cycle of F.

Now, if $n \geq 5$, we define G: $\{0,1\}^n \to \{0,1\}^n$ by setting

$$g_i(x_1, x_2, \ldots, x_n) \stackrel{\text{def}}{=} \begin{cases} f_i(x_1, x_2, x_3, x_4), & i \leq 4, \\ 0, & 4 < i \leq n. \end{cases}$$

Then $\mathbf{0}$ is a unique fixed point of G and for each x in $\{0,1\}^n$,

$$G'(x) = \begin{pmatrix} F'(x_1, x_2, x_3, x_4) & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\rho(G'(x)) = 0$ for all x in $\{0,1\}^n$. Because $\{(0,0,0,1,0,\ldots,0),$ $(1,0,1,0,0,\ldots,0),\ (1,1,0,0,\ldots,0)\}$ is a cycle of G, $\mathbf{0}$ is not a global attractor. This completes the proof of (b) of Theorem 3.2. ∎

Remark. Because there are $(2^4)^{2^4}$ maps of $\{0,1\}^4$ into itself and each map has 2^4 discrete derivatives, the example given in Theorem 3.2 has to be constructed by numerical experiments. In this connection, let us recall that a *switching function* of n variables is a function that assigns to each bit string of length n a number 0 or 1. Two switching functions $f_i(x_1, \ldots, x_n)$ and $f_j(x_1, \ldots, x_n)$ are said to be *equivalent* if there is a permutation π of $\{1, \ldots, n\}$ so that

$$f_i(x_1, \ldots, x_n) = f_j(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

By the *Pólya theory of enumeration* we can count the number of equivalence classes of switching functions. The number of equivalence classes of switching functions of four variables can be shown to be 37,333,248 (see [20], Chap. 3).

6. A CONJECTURE

The following conjecture, Conjecture 6.1, was proposed by Cima, Gasull, and Mañosas in [6]. It is a remarkable fact that Conjecture 6.1 is equivalent to the long-standing Jacobian conjecture in algebraic geometry as is proved in [6]. The Jacobian conjecture states that if F: $\mathbf{C}^n \to \mathbf{C}^n$ is a polynomial map with $\det(JF(x)) \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ at each $x \in \mathbf{C}^n$, then F is invertible.

Conjecture 6.1. If F: $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a polynomial map such that for each $x \in \mathbf{R}^n$ the spectral radius $\rho(JF(x)) < 1$, then F has a unique fixed point.

The case $n = 1$ holds trivially. Cima, Gasull, and Mañosas [6] proved that the answer to Conjecture 6.1 is affirmative for $n = 2$. Let us note the answer to Conjecture 6.1 is negative if the given polynomial map F is replaced by the $\mathscr{C}^1$ map, as the simple example $F(x) = \ln(1 + e^x)$ shows.

Conjecture 6.1 suggests the following Boolean counterpart conjecture which may be worth studying.

Conjecture 6.2. Let F: $\{0,1\}^n \longrightarrow \{0,1\}^n$. If $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$, then F has a unique fixed point.

By Theorem 3.2, the answer to Conjecture 6.2 is affirmative if $n \leq 3$. The answer to Conjecture 6.2 is also affirmative if $n = 4$, and this case $n = 4$ can be proved by the reduction method as in the proof of (a) of Theorem 3.2. However, this approach is out of the question for $n \geq 5$ because the number of *switching functions* $f_i(x_1, \ldots, x_n)$ of n variables grows astronomically as n increases. Thus the case $n \geq 5$ of Conjecture 6.2 remains open.

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